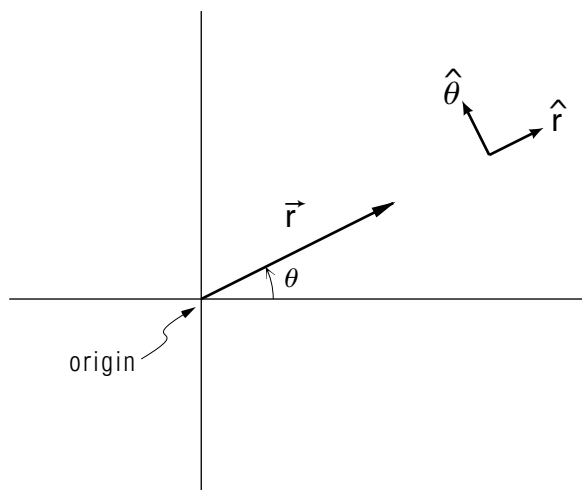


DERIVATION OF THE CONSTANTS OF THE MOTION FOR CENTRAL FORCES



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by
Peter Signell

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Input Skills:

1. State the Law of Conservation of Angular Momentum (MISN-0-41).
2. State the Law of Conservation of Energy (MISN-0-21).

Output Skills (Knowledge):

- K2. Separate the polar-coordinate variables in Newton's Second Law and thereby derive the constants of motion, angular momentum and energy, for central forces.
- K3. Derive the time derivatives of the polar coordinate unit vectors.

Post-Options:

1. "Derivation of Inverse Square Law Force Field Orbits" (MISN-0-106).

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1. Introduction

In this module the constancy of angular momentum and energy for central forces is derived from Newton's Second Law in a straightforward manner using vector algebra. Elsewhere the results of this module are used to derive the equations for planetary orbits.¹

2. Motion In a Plane

We will deal only with motion in a plane, which is the general case for motion of an object in a central force. The coordinates used will be polar, as shown in Fig. 1. Unit vectors will be denoted \hat{r} and \hat{s} as shown in Fig. 1. Thus $\vec{r} = r\hat{r}$. For motion at fixed θ , which is radial motion, both \hat{r} and \hat{s} remain constant (unchanged) as time increases. However if \hat{s} changes with time, then in time dt the unit vector \hat{r} will change by an amount $(d\theta)\hat{s}$ as shown in Fig. 2.

¹See "Derivation of Inverse Square Law Force Field Orbits" (MISN-0-106).

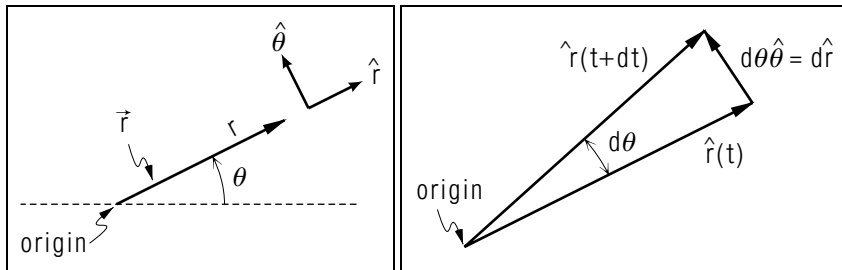


Figure 1. Polar coordinate symbols used here.

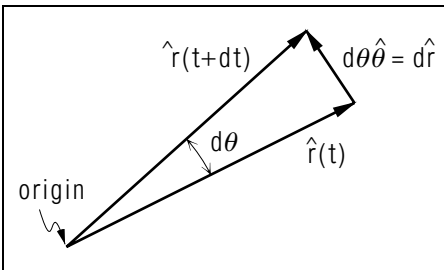


Figure 2. The radial unit vector at two times.

That is: $\hat{r}(t + dt) = \hat{r}(t) + (d\theta)\hat{s}$. Then:

$$\frac{d\hat{r}}{dt} = \frac{\hat{r}(t + dt) - \hat{r}(t)}{dt} = \frac{d\theta}{dt}\hat{s}.$$

We will henceforth use the "dot" notation for time derivatives. Then the above equation becomes:

$$\dot{\hat{r}} = \dot{\theta}\hat{s}.$$

Similarly, you can easily derive that

$$\dot{\hat{s}} = -\dot{\theta}\hat{r}.$$

Then returning to $\vec{r} = r\hat{r}$, we can easily find its derivative by taking the derivative of the product:

$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}.$$

Taking the derivative again,

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} - r\dot{\theta}^2\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s}.$$

3. Central Forces

3a. Definition. A central force is one that is completely radial:

$$\vec{F}_c = f(r, \theta)\hat{r}.$$

Then by Newton's Second Law:

$$\vec{F}_c = f(r, \theta)\hat{r} = m\vec{a} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s}. \quad (1)$$

3b. Angular Component: Conserv. of Angular Momentum.

Taking angular components by taking the dot product of both sides of Eq. (1) by \hat{s} :

$$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}).$$

In order to form a complete differential, we multiply both sides by r :

$$0 = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{d}{dt}(mr^2\dot{\theta}).$$

Thus the time derivative of $mr^2\dot{\theta}$ is zero, so

$$mr^2\dot{\theta} = \text{constant};$$

it is, in fact, the angular momentum about an axis normal (perpendicular) to the plane of motion and passing through the origin:

$$L = mr^2\dot{\theta} = I\omega = \text{constant}. \quad (2)$$

3c. Radial Components: Conservation of Energy. Taking radial components by taking the dot product of both sides of Eq. (1) with \hat{r} :

$$f(r, \theta) = m(\ddot{r} - r\dot{\theta}^2).$$

We can eliminate $\dot{\theta}$ by solving Eq. (2) for it. Then

$$f(r, \theta) = m\left(\ddot{r} - \frac{L^2}{m^2 r^3}\right) = m\ddot{r} - \frac{L^2}{mr^3}.$$

If the force function $f(r, \theta)$ is a function of radius alone, then

$$m\ddot{r} - \frac{L^2}{mr^3} - f(r) = 0.$$

In order to form a complete differential we multiply both sides by \dot{r} to get:

$$m\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} - \dot{r}f(r) = 0. \quad (3)$$

To see that this is a complete differential, form:

$$v^2 = \vec{v} \cdot \vec{v} = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{L^2}{m^2 r^2}, \quad (4)$$

so that:

$$\frac{d}{dt}v^2 = 2\dot{r}\ddot{r} - \frac{2L^2}{m^2 r^3}\dot{r}.$$

Then Eq. (3) can be written:

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) - \dot{r}f(r) = 0. \quad (5)$$

The last term can also be converted to differential form by defining:

$$E_p(r) = \int f(r) dr,$$

so that:

$$f(r) = -\frac{dE_p(r)}{dr}.$$

Then:

$$\frac{dE_p(r)}{dt} = \frac{dE_p(r)}{dr}\dot{r} = -f(r)\dot{r},$$

and Eq. (5) can be written:

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + E_p(r)\right] = 0.$$

Then:

$$\frac{1}{2}mv^2 + E_p(r) = \text{constant},$$

where $E_p(r) = -\int f(r)dr$ is the potential energy. The equation is sometimes written in the “radial plus angular kinetic energy” form, obtained by substituting Eq. (4) into Eq. (5):

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + E_p(r) = \text{constant}.$$

This, in turn, can be written in terms of radial momentum and moment of inertia as:

$$\frac{p_r^2}{2m} + \frac{L^2}{2I} + E_p(r) = \text{constant}. \quad (6)$$

Then Eq. (6) is the sum of radial kinetic, angular kinetic, and potential energy.

4. Conclusion

The appearance of Conservation of Angular Momentum from the angular equation and Conservation of Energy from the radial equation is very interesting. A whole different formulation, called the Hamiltonian formalism, uses such ideas as its basis. Identification of the constants of motion is thereby greatly simplified, especially in cases of complex motions.

Acknowledgments

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MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. Credit may be received by taking a written exam (closed book, no notes) or by means of a lecture (closed book, one 3" × 5" card of notes). For lecture credit, there must be no gaps, erasures, or pauses. Be sure to include diagrams in either exam mode.

