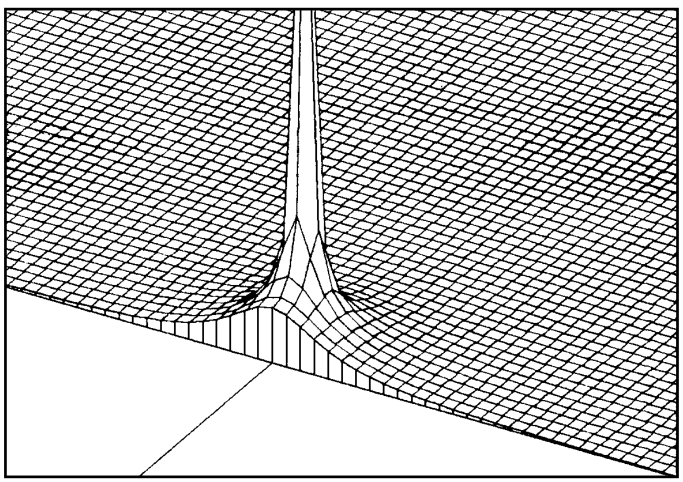
MISN-0-49

RESONANCES AND POLES; REAL AND IMAGINARY WORLDS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

RESONANCES AND POLES; REAL AND IMAGINARY WORLDS

by

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Title: **Resonances and Poles; Real and Imaginary Worlds**

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Input Skills:

1. Describe the time-average steady state power transferred into a damped driven oscillator from its driving force (MISN-0-31).
2. Plot pole trajectories of any given reciprocal of a quadratic function (MISN-0-59).

Output Skills (Knowledge):

K1. Suppose there is a narrow resonance in a physical system and state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.

Output Skills (Rule Application):

R1. Sketch complex-plane pole trajectories for given single functions of frequency.

R2. Given the observed width and position of a resonance, determine the approximate position of a nearby pole.

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1. Locating the Poles

It can be shown that the time-average steady-state power fed into a damped driven oscillator is:[1]

$$P_{\rm ave}(\omega) = \frac{F_0^2\omega_0^2\gamma/m}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\,, \quad (1)$$

where ω is the driving frequency (hence the frequency of oscillation of the oscillator), F_0 is the amplitude of the sinusoidal driving force, ω_0 is the frequency of the oscillator when undamped and undriven, and γ is the damping constant. If we consider the frequency of oscillation ω to be a complex variable, then the denominator can be factored (zeros found) by applying the quadratic root formula with ω^2 as the variable. Then the roots of the denominator, ω_p, are solutions to:

$$\omega_p^2 = -2\gamma^2+\omega_0^2 \pm 2i\sqrt{\omega_0^2-\gamma^2}\,.$$

In turn, the square root of ω_p^2 gives the actual roots:

$$\omega_p = \pm\sqrt{\omega_0^2-\gamma^2} \pm i\gamma\,. \quad (2)$$

▷ Square this to prove that it is indeed the square root.

We can now write:

$$P_{\rm ave}(\omega) = \frac{F_0^2\omega^2\,\gamma/m}{(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)}\,, \quad (3)$$

where the four ω's are the four values one obtains with the four possible sign combinations in Eq. (2). At each of these roots of the denominator the value of P becomes infinite so P is said to have a simple pole there. The pole locations have an obvious symmetry (see Fig. 1). Note that the radius vector to any pole has the length:

$$\sqrt{(\mathcal{R}e\{\omega_p\})^2+(\mathcal{I}m\{\omega_p\})^2} = \omega_0\,,$$

[1] See "Damped Driven Oscillations; Mechanical Resonances" (MISN-O-31).

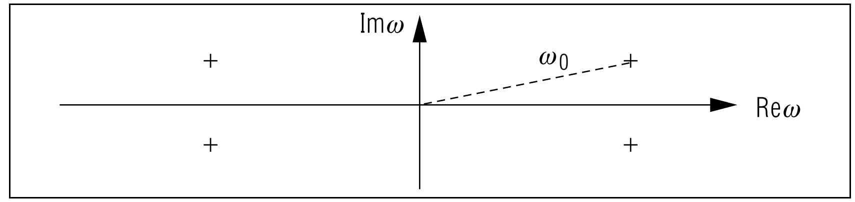

Figure 1. Pole positions are shown by +'s.

which is independent of the amount of damping in the system. Thus as you increase the damping the poles trace out trajectories which are arcs of circles of constant radius ω_0.

▷ Plot these arcs on the graph above.

▷ When you reach $\gamma = \omega_0$ the poles are all on the imaginary axis. Where do the trajectories go as you continue to increase γ beyond ω_0?

▷ What happens as $\gamma \to \infty$?

▷ How do these trajectories correlate with under damping, critical damping and overdamping?[2] We now go back to the small-damping case, $\gamma \ll \omega_0$, where the poles in the first and fourth quadrants have the positions:

$$\omega_{1,4} = \sqrt{\omega_0^2-\gamma^2} \pm \gamma \simeq \omega_0 \pm i\gamma\,.$$

In this case the real parts of these pole positions are both very close to the resonant frequency ω_0, as shown in Fig. 1.

[2] See "Damped Mechanical Oscillations" (MISN-0-29).

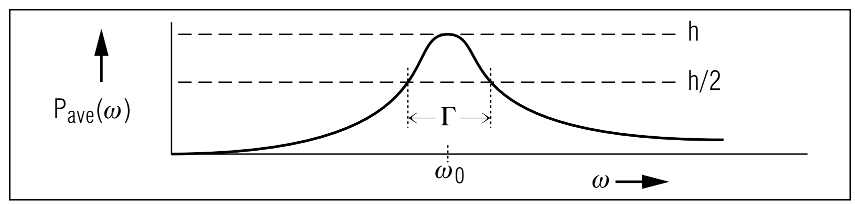

Figure 2. Definitions of resonance parameters.

2. Resonance Width

The width Γ of the resonance at half-maximum can be deduced by writing:

$$P(\omega_0 \pm \Gamma/2) = \frac{1}{2}\, P(\omega_0)\,. \tag{4}$$

The resonance parameters are illustrated in Fig. 2.

For the case $\gamma \ll \omega_0$, the resonance will turn out to have a width Γ which is very small compared to the resonant frequency ω_0 so that the denominator of $P(\omega_0 + \Gamma/2)$ can be written:

$$[\omega_0^2 - (\omega_0 \pm \Gamma/2)^2]^2 + 4\gamma^2(\omega_0 \pm \Gamma/2)^2 \simeq \omega_0^2(\Gamma^2 + 4\gamma^2)\,.$$

Putting this and $\omega^2 \simeq \omega_0^2$ into (4) yields:

$$\frac{F_0^2\omega_0^2\gamma/m}{\omega_0^2(\Gamma^2+4\gamma^2)} = \frac{1}{2}\,\frac{F_0^2\omega_0^2\gamma/m}{4\gamma^2\omega_0^2}\,.$$

for which the solution is $\gamma = \Gamma/2$. Incidentally, this result confirms that for small damping, $\gamma \ll \omega_0$, we have a narrow width: $\Gamma \ll \omega_0$.

Then the imaginary parts of the pole positions for $\gamma \ll \omega_0$ are given by the half-width at half-maximum of the observed resonance. Thus as damping is made smaller (γ smaller), the poles approach the real axis from each side and the resonance gets narrower and higher.

The Appendix shows you the case: $\gamma = 0.209\,\omega_0$.

▷ In the Appendix figures, color the $\mathcal{I}m\{\omega\}$ axis red, the vertical surface along the positive $\mathcal{R}e\{\omega\}$ axis blue.

Acknowledgments

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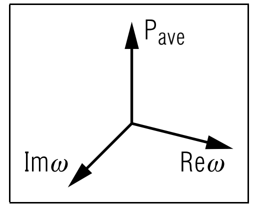

Figure 3. Definitions of the three axes for Figures 4 and 5.

A. Pictures of Poles and Resonances

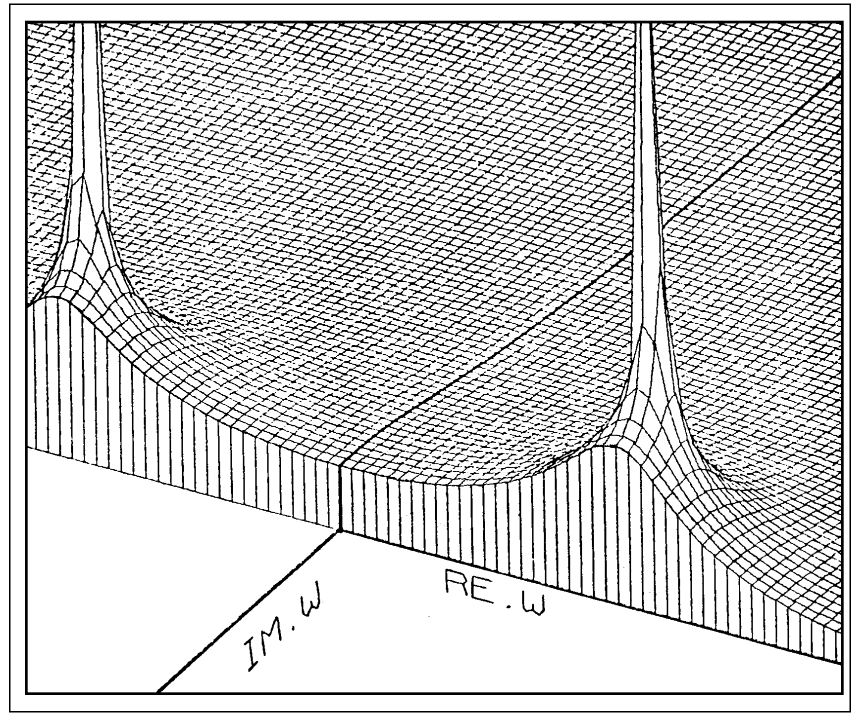

Figure 4. $P_{\rm ave}(\omega)$. The surface has been removed for $\mathcal{I}m\{\omega\} > 0$.

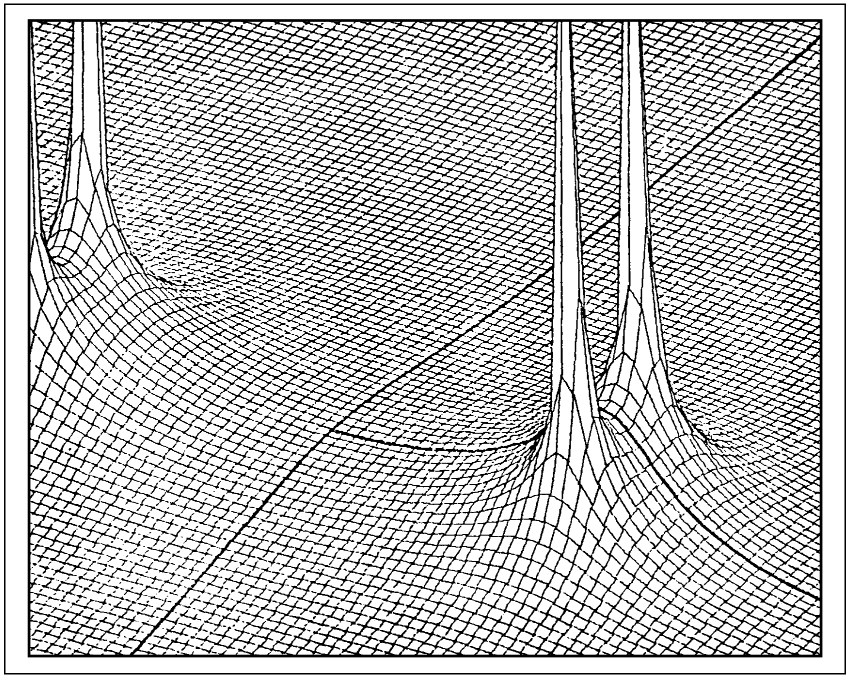

Figure 5. As in Fig. 3, but entire surface shown.

PROBLEM SUPPLEMENT

These problems also occur on this module's *Model Exam*.

1. Locate the poles of: $F(\omega) = \dfrac{a^2}{(\omega - \omega_0)^2 + a^2}$.
2. Sketch the pole trajectories resulting from variations of the parameter a in problem (1).
3. For the above case, determine the relations between the pole positions and the resonance width and position.
4. If there is a narrow resonance in a physical system, state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.

Brief Answers:

1. Solve the denominator for ω.
2. Your plot should show the two poles fleeing the real axis in opposite directions along a single straight line as a is increased.
3. You should find, good for all values of a: $\Gamma = 2\,\mathcal{I}m\{\omega_p\}$, and $\omega_{res} = \mathcal{R}e\{\omega_p\}$.
4. See this module's *text*, and think about it.

MODEL EXAM

These problems also occur in this module's *Problem Supplement*.

1. Locate the poles of: $F(\omega) = \dfrac{a^2}{(\omega - \omega_0)^2 + a^2}$.

2. Sketch the pole trajectories resulting from variations of the parameter a in problem (1).

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4. If there is a narrow resonance in a physical system, state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.