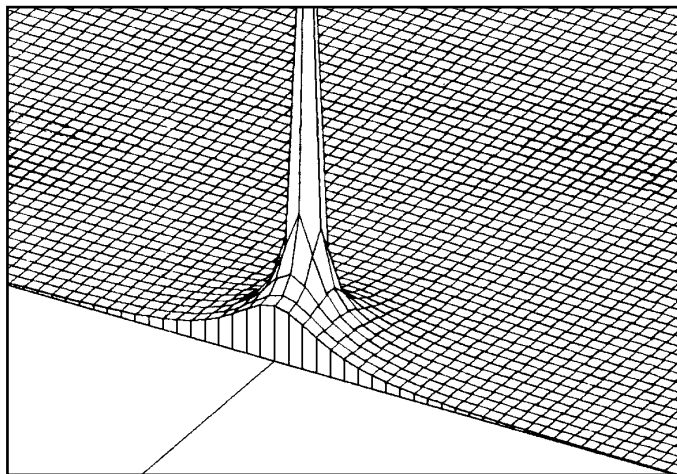


## RESONANCES AND POLES; REAL AND IMAGINARY WORLDS



RESONANCES AND POLES; REAL AND IMAGINARY WORLDS

by  
Peter Signell

1. Locating the Poles.....	1
2. Resonance Width.....	3
Acknowledgments.....	3
A. Pictures of Poles and Resonances.....	4

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**Input Skills:**

1. Describe the time-average steady state power transferred into a damped driven oscillator from its driving force (MISN-0-31).
2. Plot pole trajectories of any given reciprocal of a quadratic function (MISN-0-59).

**Output Skills (Knowledge):**

- K1. Suppose there is a narrow resonance in a physical system and state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.

**Output Skills (Rule Application):**

- R1. Sketch complex-plane pole trajectories for given single functions of frequency.
- R2. Given the observed width and position of a resonance, determine the approximate position of a nearby pole.

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# RESONANCES AND POLES; REAL AND IMAGINARY WORLDS

by  
Peter Signell

## 1. Locating the Poles

It can be shown that the time-average steady-state power fed into a damped driven oscillator is:<sup>1</sup>

$$P_{\text{ave}}(\omega) = \frac{F_0^2 \omega_0^2 \gamma / m}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2}, \quad (1)$$

where  $\omega$  is the driving frequency (hence the frequency of oscillation of the oscillator),  $F_0$  is the amplitude of the sinusoidal driving force,  $\omega_0$  is the frequency of the oscillator when undamped and undriven, and  $\gamma$  is the damping constant. If we consider the frequency of oscillation  $\omega$  to be a complex variable, then the denominator can be factored (zeros found) by applying the quadratic root formula with  $\omega^2$  as the variable. Then the roots of the denominator,  $\omega_p$ , are solutions to:

$$\omega_p^2 = -2\gamma^2 + \omega_0^2 \pm 2i\sqrt{\omega_0^2 - \gamma^2}.$$

In turn, the square root of  $\omega_p^2$  gives the actual roots:

$$\omega_p = \pm \sqrt{\omega_0^2 - \gamma^2} \pm i\gamma. \quad (2)$$

▷ Square this to prove that it is indeed the square root.

We can now write:

$$P_{\text{ave}}(\omega) = \frac{F_0^2 \omega^2 \gamma / m}{(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)}, \quad (3)$$

where the four  $\omega$ 's are the four values one obtains with the four possible sign combinations in Eq. (2). At each of these roots of the denominator the value of  $P$  becomes infinite so  $P$  is said to have a simple pole there. The pole locations have an obvious symmetry (see Fig. 1). Note that the radius vector to any pole has the length:

$$\sqrt{(\text{Re}\{\omega_p\})^2 + (\text{Im}\{\omega_p\})^2} = \omega_0,$$

<sup>1</sup>See “Damped Driven Oscillations; Mechanical Resonances” (MISN-O-31).

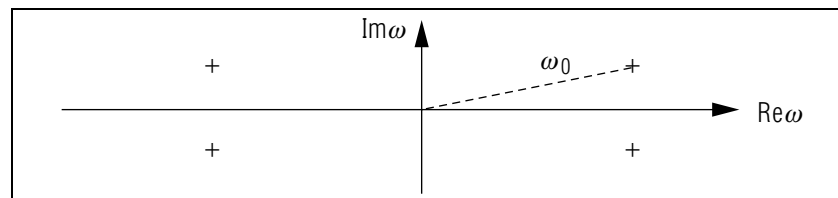


Figure 1. Pole positions are shown by '+'s.

which is independent of the amount of damping in the system. Thus as you increase the damping the poles trace out trajectories which are arcs of circles of constant radius  $\omega_0$ .

▷ Plot these arcs on the graph above.

▷ When you reach  $\gamma = \omega_0$  the poles are all on the imaginary axis. Where do the trajectories go as you continue to increase  $\gamma$  beyond  $\omega_0$ ?

▷ What happens as  $\gamma \rightarrow \infty$ ?

▷ How do these trajectories correlate with under damping, critical damping and overdamping?<sup>2</sup> We now go back to the small-damping case,  $\gamma \ll \omega_0$ , where the poles in the first and fourth quadrants have the positions:

$$\omega_{1,4} = \sqrt{\omega_0^2 - \gamma^2} \pm \gamma \simeq \omega_0 \pm i\gamma.$$

In this case the real parts of these pole positions are both very close to the resonant frequency  $\omega_0$ , as shown in Fig. 1.

<sup>2</sup> See “Damped Mechanical Oscillations” (MISN-0-29).

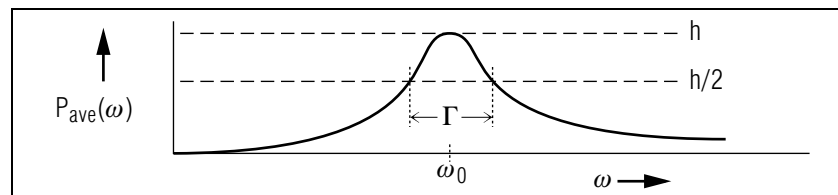


Figure 2. Definitions of resonance parameters.

## 2. Resonance Width

The width  $\Gamma$  of the resonance at half-maximum can be deduced by writing:

$$P(\omega_0 \pm \Gamma/2) = \frac{1}{2} P(\omega_0). \quad (4)$$

The resonance parameters are illustrated in Fig. 2.

For the case  $\gamma \ll \omega_0$ , the resonance will turn out to have a width  $\Gamma$  which is very small compared to the resonant frequency  $\omega_0$  so that the denominator of  $P(\omega_0 + \Gamma/2)$  can be written:

$$[\omega_0^2 - (\omega_0 \pm \Gamma/2)^2]^2 + 4\gamma^2(\omega_0 \pm \Gamma/2)^2 \simeq \omega_0^2(\Gamma^2 + 4\gamma^2).$$

Putting this and  $\omega^2 \simeq \omega_0^2$  into (4) yields:

$$\frac{F_0^2 \omega_0^2 \gamma / m}{\omega_0^2(\Gamma^2 + 4\gamma^2)} = \frac{1}{2} \frac{F_0^2 \omega_0^2 \gamma / m}{4\gamma^2 \omega_0^2}.$$

for which the solution is  $\gamma = \Gamma/2$ . Incidentally, this result confirms that for small damping,  $\gamma \ll \omega_0$ , we have a narrow width:  $\Gamma \ll \omega_0$ .

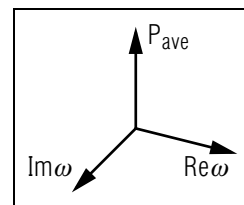
Then the imaginary parts of the pole positions for  $\gamma \ll \omega_0$  are given by the half-width at half-maximum of the observed resonance. Thus as damping is made smaller ( $\gamma$  smaller), the poles approach the real axis from each side and the resonance gets narrower and higher.

The Appendix shows you the case:  $\gamma = 0.209 \omega_0$ .

▷ In the Appendix figures, color the  $\mathcal{I}m\{\omega\}$  axis red, the vertical surface along the positive  $\mathcal{R}e\{\omega\}$  axis blue.

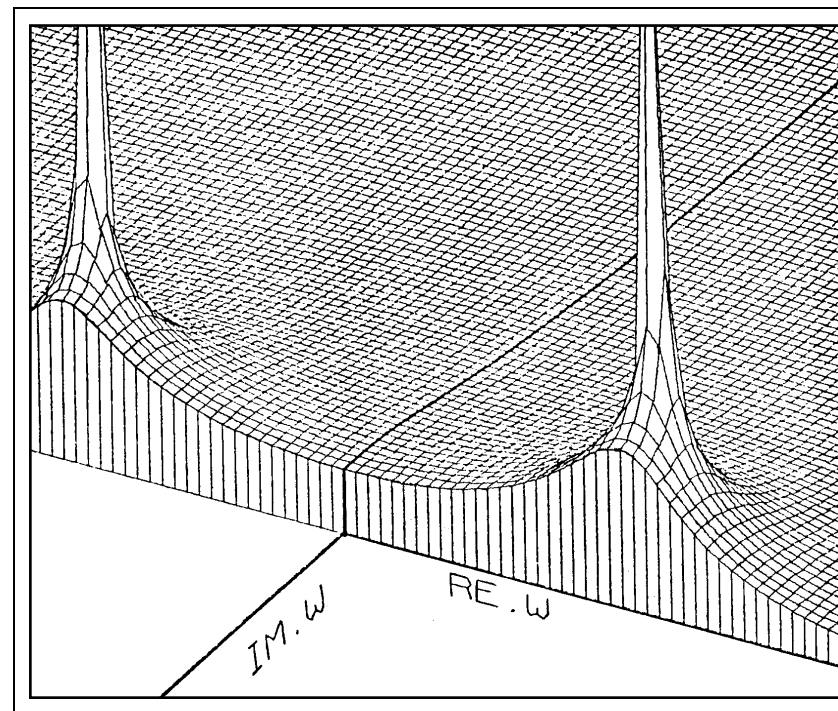
## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.



**Figure 3.** Definitions of the three axes for Figures 4 and 5.

## A. Pictures of Poles and Resonances

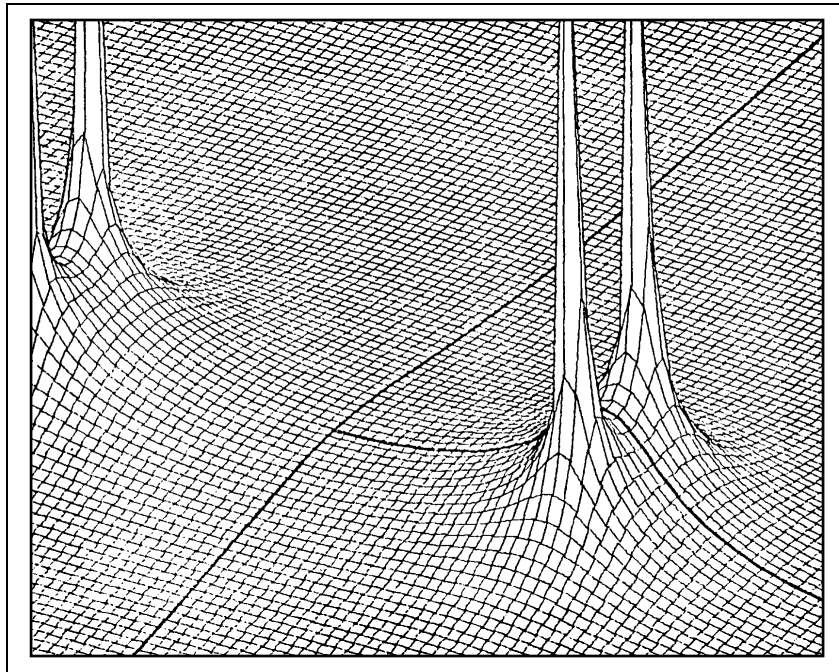


**Figure 4.**  $P_{\text{ave}}(\omega)$ . The surface has been removed for  $\mathcal{I}m\{\omega\} > 0$ .

## PROBLEM SUPPLEMENT

These problems also occur on this module's *Model Exam*.

1. Locate the poles of:  $F(\omega) = \frac{a^2}{(\omega - \omega_0)^2 + a^2}$ .
2. Sketch the pole trajectories resulting from variations of the parameter  $a$  in problem (1).
3. For the above case, determine the relations between the pole positions and the resonance width and position.
4. If there is a narrow resonance in a physical system, state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.



**Figure 5.** As in Fig. 3, but entire surface shown.

### Brief Answers:

1. Solve the denominator for  $\omega$ .
2. Your plot should show the two poles fleeing the real axis in opposite directions along a single straight line as  $a$  is increased.
3. You should find, good for all values of  $a$ :  $\Gamma = 2\mathcal{I}m\{\omega_p\}$ , and  $\omega_{res} = \mathcal{R}e\{\omega_p\}$ .
4. See this module's *text*, and think about it.

**MODEL EXAM**

These problems also occur in this module's *Problem Supplement*.

1. Locate the poles of:  $F(\omega) = \frac{a^2}{(\omega - \omega_0)^2 + a^2}$ .
2. Sketch the pole trajectories resulting from variations of the parameter  $a$  in problem (1).
3. For the above case, determine the relations between the pole positions and the resonance width and position.
4. If there is a narrow resonance in a physical system, state what measurements you could make in order to determine the approximate locations of the nearby poles. State the conditions under which the approximate locations are accurate.