PHYSNET

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BOUNDARY CONDITIONS: VIBRATING STRINGS, HEAT DIFFUSION

Math Physics

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BOUNDARY CONDITIONS: VIBRATING STRINGS, HEAT DIFFUSION

by

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Title: **Boundary Conditions: Vibrating Strings, Heat Diffusion**

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Input Skills:

1. Vocabulary: periodic function, Fourier expansion of $f(x)$, orthogonal function, Fourier integral expansion of f(x), Fourier transform of $f(x)$, Fourier integral theorem, partial differential equation.
2. Unknown: assume (MISN-0-485).

Output Skills (Knowledge):

K1. Vocabulary: partial differential equation, order of P.D.E., solution of P.D.E., boundary conditions, boundary value problem, linear and nonlinear P.D.E., elliptic P.D.E., hyperbolic P.D.E., parabolic P.D.E., diffusion and heat conducting equation, vibrating string equation, Laplace's equation, Poisson's equation, Dirichlet boundary conditions, Neumann boundary conditions, Cauchy boundary conditions.

K2. Classify a given P.D.E. according to these characteristics: linear or nonlinear, homogeneous or nonhomogeneous, order, elliptic, hyperbolic or parabolic.

K3. Derive the diffusion equation.

Output Skills (Rule Application):

R1. Write down the mathematical B.V.P. corresponding to a physical problem when given the problem in words.

R2. Compute the solution to a physical problem involving vibrating strings with ends clamped or temperatures in a slab or bar.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

In this unit, you will come face to face with two very important applications of Fourier series and integrals in the solution of problems involving vibrating systems and heat diffusion. Partial differential equations of various types will be introduced and analyzed first. Then, the diffusion equation is derived in order to illustrate how a physical problem results in such a partial differential equation. Finally, solutions to boundary value problems involving vibrating strings and heat diffusion are obtained. Since this unit has a rather extensive reading assignment, I will cut off my introductory remarks here so that you can get started on the unit.

2. Procedures

1. a. Read pages 258-260 of Spiegel.
   b. The question of appropriate boundary conditions in a physical problem is a touchy one in which you can only gain skill by experience. In order to give you a conceptual framework to use in your future work read Section 1, page 9-3 to 9-7 of the Supplementary Notes.
2. Write down or underline in the text each of the definitions and concepts of Output Skill K1.
3. Read through the Solved Problems 12.1 and 12.2 on "Classification of Partial Differential Equations" in Spiegel, page 261.
4. Solve Supplementary Problems 12.29 and 12.30 on "Classification of Partial Differential Equations" in Spiegel, pages 277-278.
5. Write down the derivation of the diffusion equation as given in Section II, page 9-7 to 9-9, of the Supplementary Notes. You will be asked to write down this derivation from memory on the Unit Test. WARNING: Although I have used the word *memory* here, the most

efficient way to remember a derivation is to *understand* the concepts involved.
6. Read through Solved Problems 12.6, page 263, of Spiegel. This problem gives an alternative derivation of the source-free diffusion equation.
7. Read through Solved Problems 12.6, pages 266-267, of Spiegel on solving a heat diffusion problem using separation of variables.
8. Read through Sections III to V, pages 9-9 to 9-20, of the Supplementary Notes. These sections treat the solution of B. V. P. involving vibrating strings and heat diffusion.
9. Read through Solved Problems 12.17, 12.18, 12.19, 12.21, and 12.23 of Spiegel on B. V. P. involving vibrating strings and heat diffusion.
10. Solve the Supplementary Problems 12.29, 12.30, 12.48, 12.50, 12.53, 12.65, and 12.66* of Spiegel on B. V. P. involving vibrating strings and heat diffusion. You may simply apply the appropriate boxed equations from the Supplementary Notes.

Note: You are allowed one sheet of $8.5'' \times 11.0''$ paper containing the boxed equations in the Supplementary Notes when taking the Unit Test. The sheet can also contain the B. V. P. for which the boxed equations are solutions.

* Correct answer is:

$$u(x,y) = \frac{\mu_0}{2} + \frac{\mu_0}{\pi}\tan^{-1}\frac{x}{y}\,.$$

3. Supplementary Notes

3a. Boundary Conditions. A physical problem is not uniquely specified if the partial differential equation which the solution of the problem must satisfy is given. There is an infinite number of solutions of the partial differential equations listed on pages 259 and 260 of Spiegel. Any physical problem must state not only the partial differential equation which is to be solved but also the *boundary conditions* which the solution must satisfy. In fact, satisfying of the boundary conditions (B. C.) is often as difficult a task as the solving of the partial differential equation (P. D. E.).

However, the solutions of a given P. D. E. cannot be made to satisfy any sort of B. C. For each type of P. D. E. . on pages 259 and 260 of Spiegel, there is a definite set of B. C. which will give unique answers. An actual physical problem will always have the right sort of B. C. to give it a unique answer (we hope!), and if the statement of the problem corresponds to reality, the right B. C. are guaranteed. But it is not always easy to tell just what B. C. correspond to "reality." Thus, the types of B. C. which are suitable for various P. D. E. are given below. This can guide a scientist or engineer in making his mathematical problem fit the physical problem as closely as possible. The best and most accessible reference on B. C. in P. D. E. remains Morse and Feshbach, *Methods of Theoretical Physics*, Sec. 6.1, pages 676-692, McGraw-Hill (1953).

There are essentially four types of B. C. In order to make the definitions precise, let the P. D. E. be written

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2}{\partial y^2}=H(x,y,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) \tag{1}$$

where

$$H(x,y)=D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial x}+F(x,y)u-G(x,y) \tag{2}$$

The functions A, B, C, D, E, F, and G are specified functions of x and y in some domain R bounded by a curve S. The two variables x and y may be either two space coordinates or one space and one time coordinate. (If there are three spatial dimensions involved, R is a volume V and S is a surface). From this point of view, initial conditions are boundary conditions in time. Then the four types of B. C. are:

1. Dirichlet B. C. The value of u is specified on S.
2. Neumann B. C. The value of the normal derivative $\partial u/\partial n$ is specified on S.
3. Cauchy B. C. The values of u and $\partial u/\partial n$ are specified on S.
4. Mixed B. C. Some combination of the above three types of B. C. are given for various portions on S.

The P. D. E. are classified as in Spiegel on page 258:

1. Elliptic equations have $B^2-4AC<0$. Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{3}$$

and Poisson's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-4\pi\rho(x,y) \tag{4}$$

are examples.

2. Hyperbolic equations have $B^2-4AC>0$. The wave equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}=0 \tag{5}$$

is an example.

3. Parabolic equations have $B^2-4AC=0$. The diffusion equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{\kappa}\frac{\partial u}{\partial t}=0 \tag{6}$$

is an example.

As an example, the solution of Poisson's equation inside a volume V subject to either Dirichlet or Neumann B. C. on the closed surface S is unique. To show this, assume that there are two functions u_1 and u_2 such that

$$\nabla^2 u_1=-4\pi\rho$$

and

$$\nabla^2 u_2=-4\pi\rho \tag{7}$$

where either u_1 and u_2 are given an S (where $u_1=u_2$ on S) corresponding to Dirichlet B. C. or $\partial u_1/\partial n$ and $\partial u_2/\partial n$ are given on S (where

$$\frac{\partial u_1}{\partial n}=\frac{\partial u_2}{\partial n} \tag{8}$$

on S) corresponding to Neumann B. C. Then define $U=u_1-u_2$. Now, $U=0$ on S if Dirichlet B. C. hold, or $\partial U/\partial n=0$ on S if Neumann B. C. hold. Applying eq. (1.101) of Arfken gives

$$\int_V (U\nabla^2 U+\vec{\nabla}U\cdot\vec{\nabla}U)\,d\tau=\int_V \vec{\nabla}\cdot(U\vec{\nabla}U)\,d\tau \tag{9}$$

$$= \int_S U\vec{\nabla}U \cdot d\vec{\sigma}\,.$$

So,

$$\int_V (U\nabla^2 U + \vec{\nabla}U \cdot \vec{\nabla}U)\, d\tau = \int_S U \frac{\partial U}{\partial n}\, dS \tag{10}$$

where $d\vec{\sigma} = \hat{n}\, dS$ and $\vec{\nabla}U \cdot \hat{n} = \partial U/\partial n$. Also, $\nabla^2 U = 0$.

But, for either type of B. C., the surface integral vanishes, so that

$$\int_V |\vec{\nabla}U|^2 = 0\,. \tag{11}$$

This last equation will be true for any volume V so that

$$\vec{\nabla}U = 0 \tag{12}$$

inside V. Thus,

$$U = \text{ constant} \tag{13}$$

inside V. In the case of Dirichlet B. C., this constant is zero so that $u_1 = u_2$. In the case of Neumann B. C., this constant may not be zero so u_1 and u_2 only differ by an unimportant arbitrary, additive constant.

It should be clear that a solution to Poisson's equation with Cauchy B. C. on a closed boundary (both u and $\partial u/\partial n$ specified) does not exist, since there are unique solutions for Dirichlet and Neumann B. C. Also, there is a unique solution to the problem in the case of mixed B. C. where Dirichlet B. C. hold over a portion of S and Neumann B. C. hold over the remaining part. Morse and Feshbach study the various P. D. E. and B. C. in order to come up with the following table for the uniqueness of solutions to P. D. F. with various B. C.

3b. Type of Equation. The Table:

Type of B. C.	Elliptic (Poisson's eq.)	Hyperbolic (wave eq.)	Parabolic (heat-cond. eq.)
Dirichlet			
Open surface	Not enough	Not enough	*Unique, stable sol. in one direction*
Closed surface	*Unique, stable solution*	Too much	Too much
Neumann			
Open Surface	Not enough	Not enough	*Unique, stable sol. in one direction*
Closed surface	*Unique, stable sol. in general*	Too much	Too much
Cauchy			
Open Surface	Unphys. results	*Unique, stable solution*	Too much
Closed surface	Too much	Too much	Too much

The application of this table can sometimes be not so straightforward, so a reading of Morse and Feshbach would be of great help. For example, consider the wave equation which is the prototype example of an hyperbolic equation. According to the table, a unique solution exists for the case of Cauchy B. C. on an open surface. And, indeed, this is true. That is, a unique solution exists for the boundary value problem (B. V. P.)

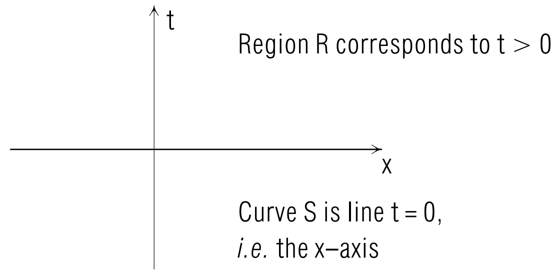
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \;\; (-\infty < x < \infty,\; 0 < t < \infty) \tag{14}$$

$$u(x,0) = f(x), \;\; (-\infty < x < \infty) \tag{15}$$

and

$$\frac{\partial u(x,0)}{\partial t} = g(x), \;\; (-\infty < x < \infty)\,. \tag{16}$$

The functions $g(x)$ and $f(x)$ are given. This amounts to solving the P. D. E. with Cauchy B. C. on an open surface where $f(x)$ is the initial displacement and $g(x)$ is the initial velocity. The open surface (curve in this case) is just the x-axis since there is no closure on t at infinity.



The proof that a unique solution exists is easy, but will not be done here. See Morse and Feshbach, page 685.

On the other hand, suppose the waving medium (say a string) is clamped at $x = 0$ and $x = L$. Then, we have these B. V. P. :

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad (0 < x < L,\ 0 < t < \infty) \tag{17}$$

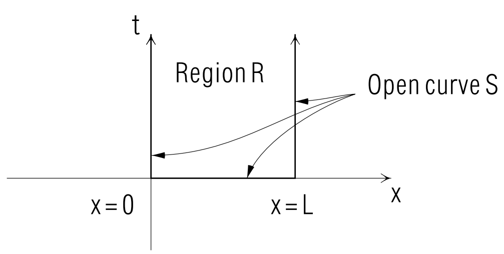
$$u(0,t) = u(L,t) = 0, \quad (0 \le t < \infty) \tag{18}$$

$$u(x,0) = f(x), \quad (0 \le x \le L) \tag{19}$$

and

$$\frac{\partial u(x,0)}{\partial t} = g(x), \quad (0 \le x \le L)\,. \tag{20}$$

The open curve S now corresponds to the lines $x = 0$, $x = L$, and $t = 0$ and is U-shaped.



The B. C. given in eqs. (19) and (20) amount to Cauchy B. C. on $t = 0$ for $0 \le x \le L$, but the B. C. in eq. (18) amounts to Dirichlet conditions on $x = 0$ and $x = L$ for $t > 0$. This occurs because the extra constraint on the problem due to the clamped ends would result in the

problem being over- defined if Cauchy B. C. were given on $x = 0$ and $x = L$. The exact reasons for this can be found in Morse and Feshbach, page 686.

3c. Derivation of the Diffusion Equation. As an example of how a physical model can result in one of the P. D. E. which are listed in section 8.1 of Arfken and on pages 259-260 of Spiegel, the diffusion equation will be derived. The derivation applies to some physical quantity $Q(t)$ of a substance. The amount of Q per unit volume is $q(\vec{r},t)$. The quantity $q(\vec{r},t)$ is a time-dependent scalar field. Therefore, the amount $Q(t)$ in volume V is given by

$$Q(t) = \int_V q(\vec{r},t)\, d\tau\,. \tag{21}$$

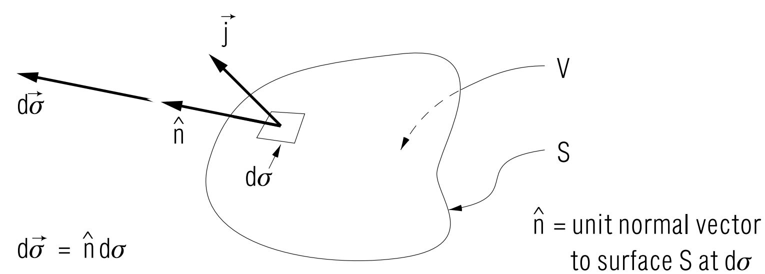
The amount of $Q(t)$ can change by two processes.

a) Flow of the physical quantity via a current. This flow is described by a current vector $\vec{j}(\vec{r},t)$ which is a vector field. For example, Q can be electric charge, q can be charge density, and $\vec{j}$ can be electric current.

b) Sources of quantity Q in volume V. The source (or sink) is described by a scalar field $\phi(\vec{r},t)$ which is the rate at which $Q(t)$ is created per unit volume per unit time.

Thus,

$$\frac{dQ}{dt} = \int_V \frac{\partial q(\vec{r},t)}{\partial t}\, d\tau = -\int_S \vec{j}\cdot d\vec{\sigma} + \int_V \phi(\vec{r},t)\, d\tau\,. \tag{22}$$



From the orientation of $\vec{j}$ and $d\vec{\sigma}$, if $-\vec{j}\cdot d\vec{\sigma}$ is a positive quantity, then it represents the flow into V across $d\vec{\sigma}$.

But

$$\int_V \vec{\nabla}\cdot\vec{j}\,d\tau = \int_S \vec{j}\cdot d\vec{\sigma}\,. \tag{23}$$

So

$$\int_V \left(\frac{\partial q}{\partial t} + \vec{\nabla}\cdot\vec{j} - \phi\right) d\tau = 0\,. \tag{24}$$

This equation holds for any volume V. Therefore,

$$\frac{\partial q}{\partial t} + \vec{\nabla}\cdot\vec{j} = \phi\,. \tag{25}$$

This is the *equation of continuity* with sources. It is essentially a conservation equation.

In some cases an empirical relation exists between $\vec{j}$ and q. Many times it is true that a gradient of q results in a flow of quantity q from higher to lower concentrations. Then

$$\vec{j}(\vec{r},t) = -x\vec{\nabla}q(\vec{r},t)\,. \tag{26}$$

If q represents charge, then $\vec{j}$ represents the current density. If q is temperature; then $\vec{j}$ represents heat flow per unit area per unit time. In the case of heat flow, this last equation is the Fourier law of heat conduction. If q represents the concentration of a solute in a solvent, then the last equation is called Fick's law. Substitution of $\vec{j}$ in eq. (26) into the equation of continuity, eq. (25), gives

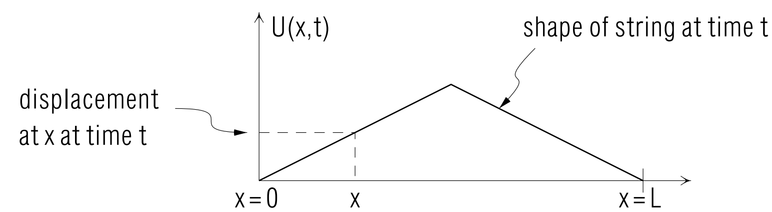
$$\frac{\partial q}{\partial t} - \vec{\nabla}\cdot(\kappa\vec{\nabla}q) = \phi\,. \tag{27}$$

If n is constant and there are no sources or sinks of the quantity (i.e. $\phi = 0$), then

$$\nabla^2 q - \frac{1}{\kappa}\frac{\partial q}{\partial t} = 0\,. \tag{28}$$

This is the heat conduction equation as given on page 259 of Spiegel.

3d. Solution of the Vibrating String with Ends Clamped. Consider the case of a string vibrating with ends clamped. Let the coordinate along the string be x, and the time coordinate be t. Let v be the speed of propagation of waves on the string.



Let $u(x,t)$ represent the displacement of the string. Then the B. V. P. can be written

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad (0 < t < \infty,\ 0 < x < L) \tag{29}$$

$$u(0,t) = u(L,t) = 0, \quad (0 \le t < \infty) \tag{30}$$

$$u(x,0) = f(x), \quad (0 \le x \le L) \tag{31}$$

and

$$\frac{\partial u(x,0)}{\partial t} = g(x), \quad (0 \le x \le L)\,. \tag{32}$$

The known functions $f(x)$ and $g(x)$ are the initial displacement and velocity of the string, respectively. I shall use the method of separation of variables to solve this problem. So assume

$$u(x,t) = X(x)T(t)\,. \tag{33}$$

Then, substitution of eq. (33) in eq. (29) yields this result:

$$X''T - \frac{1}{v^2}XT'' = 0 \tag{34}$$

or

$$\frac{X'}{'}(x)X(x) = \frac{T''(t)}{v^2 T(t)}\,. \tag{35}$$

The left-hand side of the above equation is a function of x alone so it cannot vary with t. The right-hand side is a function of t alone so it cannot vary with x. Hence, both sides must be equal to some constant, say γ, so that

$$X''(x) - \gamma X(x) = 0 \tag{36}$$

$$T''(t) - \gamma v^2 T(t) = 0\,. \tag{37}$$

From the B. C. in eq. (30), this result is obtained:

$$X(0) = X(L) = 0\,. \tag{38}$$

Solution to eq. (36) can be written

$$X(x) = C_1 e^{x\sqrt{\gamma}} + C_2 e^{-x\sqrt{\gamma}} \tag{39}$$

where C_1 and C_2 are constants. Eqs. (38) and (39) give

$$C_1 + C_2 = 0 \tag{40}$$

and

$$C_1 e^{L\sqrt{\gamma}} + C_2 e^{-L\sqrt{\gamma}} = 0\,. \tag{41}$$

Suppose $\gamma > 0$. Then

$$e^{L\sqrt{\gamma}} - e^{-L\sqrt{\gamma}} = 0 \tag{42}$$

so that

$$\sinh L\sqrt{\gamma} = 0\,. \tag{43}$$

But, the hyperbolic sine is always nonzero when $L\sqrt{\gamma} > 0$. Thus, solutions for $X(x)$ only exist when $\gamma \le 0$. Let

$$\gamma = -\beta^2, \ \ \beta^2 \ge 0\,. \tag{44}$$

Then, the expression for $X(x)$ in eq. (39) can be rearranged to give

$$X(x) = A\cos\beta x + B\sin\beta x \tag{45}$$

where A and B depend on C_1 and C_2. The B. C. in eq. (38) yield

$$A = 0 \tag{46}$$

and

$$B\sin\beta L = 0\,. \tag{47}$$

Thus

$$\beta L = n\pi, \ \ n = 1, 2, 3, \ldots \tag{48}$$

The solution $X(x)$ are actually denumberably infinite, and they can be labeled by the integer n:

$$X_n(x) = B_n \sin\frac{n\pi x}{L}, \ \ n = 1, 2, 3, \ldots \tag{49}$$

In the same way the solution $T(t)$ depends on n and can be written as

$$T_n(t) = C_n\cos\frac{n\pi vt}{L} + D_n\sin\frac{n\pi vt}{L} \tag{50}$$

where C_n and D_n are constants. The full solution $u(x,t)$ must be a linear combination of the X_nT_n, because the remaining two B. C. need to be satisfied. So, if $C'_n = B_nC_n$ and $D'_n = B_nD_n$,

$$u(x,t) = \sum_{n=1}^{\infty}\left[C'_n\cos\frac{n\pi vt}{L} + D'_n\sin\frac{n\pi vt}{L}\right]\sin\frac{n\pi x}{L}\,. \tag{51}$$

But

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C'_n\sin\frac{n\pi x}{L} \tag{52}$$

and

$$\frac{\partial u(x,0)}{\partial t} = g(x) = \frac{\pi x}{L}\sum_{n=1}^{\infty} nD'_n\sin\frac{n\pi x}{L}\,. \tag{53}$$

Remember that $0 \le x \le L$. The above two equations are Fourier sine series on $(0,L)$. Thus, these expressions can be written down immediately for C'_n and D'_n:

$$C'_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi v}{L}\,dx \tag{54}$$

and

$$\frac{n\pi v}{L}D'_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi v}{L}\,dx\,. \tag{55}$$

Eqs. (51), (54), (55) are a complete solution to the original B. V. P. Numerous examples are discussed in Spiegel on the pages indicated in the Procedures. As an example, suppose

$$\begin{aligned} f(x) &= x, \text{ when } 0 \le x \le 1 \\ &= -x + 2 \text{ when } 1 \le x \le 2 \end{aligned} \tag{56}$$

and

$$g(x) = 0 \text{ when } 0 \le x \le 2\,. \tag{57}$$

Then

$$C'_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2(-x+2)\sin\frac{n\pi x}{2}\,dx$$

so

$$C'_n = \frac{8}{\pi^2 n^2}\sin\frac{n\pi}{2}\,. \tag{58}$$

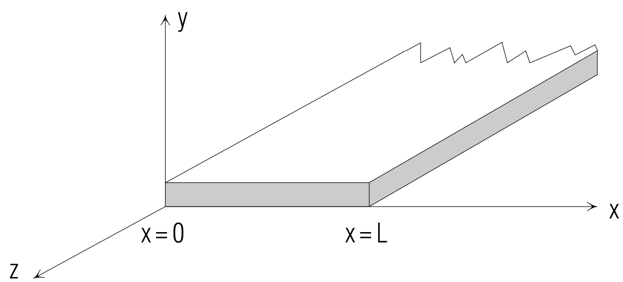
Also

$$D'_n = 0\,. \tag{59}$$

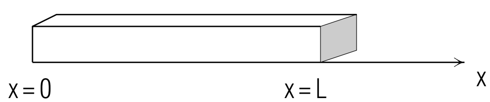
Thus

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}\cos\frac{n\pi v t}{2}/n^2, \quad (0 \le x \le 2)\,. \tag{60}$$

3e. Temperature in a Slab or Bar. Suppose that you are interested in the temperatures in a slab of homogeneous material bounded by the planes $x = 0$ and $x = L$. The dimension in the y-direction can be negligibly small and the slab can go to infinity in the z-direction.



The temperature in such a slab is only dependent on the coordinate x (and t) if the faces at $x = 0$ and $x = L$ are held at the same temperature for all z. Alternatively, the formulation will be the same for temperatures in a bar of length L where the lateral dimensions are very small.



The $u(x,t)$ represent the temperature in such a system. Then, the temperature satisfies the diffusion equation where n is the diffusivity. As seen in the table in Sec. I on B. C., the diffusion equation has a unique solution for an open boundary with either Dirichlet or Neumann boundary conditions. There could also be a mixed B. C. with Dirichlet B. C. over part of the boundary and Neumann over the remaining part. The openness of boundary is guaranteed by the lack of closure in t at infinity ($0 \le t < \infty$). Dirichlet B. C. at $x = 0$ and $x = L$, namely, $u(0,t) = g(t)$ and $u(L,t) = h(t)$ where g and h are known functions of time, correspond to knowing the temperature at the faces $x = 0$ and $x = L$. Of course, Neumann B. C. at $x = 0$ and $x = L$, that is, having $\partial u/\partial t$ at $x = 0$

and $x = L$, correspond to knowing the rate at which the temperature is changing. A usual case is to have one of the following cases:

1. $u(0,t)$ & $u(L,t)$ being constant, corresponding to constant temperature at the faces.
2. $\partial u(0,t)/\partial t$ & $\partial u(L,t)/\partial t$ being zero corresponding to thermally insulated faces .

Case 1: Let us consider this B. V. P :

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0, \quad (0 < x < L,\ 0 < t < \infty) \tag{61}$$

where

$$u(0,t) = T_0, \quad (0 \le t < \infty) \tag{62}$$

$$u(L,t) = T_1, \quad (0 \le t < \infty) \tag{63}$$

$$u(x,0) = f(x), \quad (0 \le x \le L)\,. \tag{64}$$

The function $f(x)$ is given and is the initial temperature. Redefine the temperature as

$$v(x,t) = u(x,t) - T_0 - (T_1 - T_0)\frac{x}{L}\,. \tag{65}$$

Then the new function $v(x,t)$ satisfies this B. V. P. :

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \quad (0 < x < L,\ 0 < t < \infty) \tag{66}$$

$$v(0,t) = v(L,t) = 0, \quad (0 \le t < \infty) \tag{67}$$

and

$$v(x,0) = f(x) - T_0\left(\frac{L-x}{L}\right) - T_1\frac{x}{L}, \quad (0 \le x \le L)\,. \tag{68}$$

The new temperature $v(x,t)$ has zero temperature at the faces. This makes the solution somewhat simpler. Of course, either of T_0 or T_1, can be zero. The solution to the original B. V. P., namely, $u(x,t)$, can be obtained from eq. (68) after $v(x,t)$ is obtained from the B. V. P. in eqs. (66), (67), and (68). The technique of separation of variables can be used. Assume

$$v(x,t) = X(x)T(t)\,. \tag{69}$$

Then, substitution of eq. (69) into eq. (66) yields

$$\frac{X''}{X} = \frac{T''}{\kappa T} \,. \tag{70}$$

Using the usual argument gives

$$X'' - \alpha\, X = 0 \tag{71}$$

and

$$T' - \alpha n T = 0, \quad \alpha = \text{ constant} \,. \tag{72}$$

The B. C. in eqs. (67) give

$$X(0) = X(L) = 0 \,. \tag{73}$$

Thus, the solution of eq. (71) can be written as

$$X_n(x) = B_n \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots \tag{74}$$

where

$$\alpha = -\frac{n^2\pi^2}{L^2} \,. \tag{75}$$

Then, the solution of eq. (72) can be expressed as

$$T_n(t) = C_n e^{-n^2\pi^2\kappa t/L^2} \,. \tag{76}$$

As in the previous case of the wave equation, the solution to the diffusion equation can be written as

$$v(x,t) = \sum_{n=1}^{\infty} B'_n e^{-n^2\pi^2\kappa t/L^2} \sin \frac{n\pi x}{L}, \quad B'_n = B_n C_n \,. \tag{77}$$

The last B. C. in eq. (68) can be satisfied by

$$v(x,0) = f(x) - T_0 \left(\frac{L-x}{L}\right) - T_1 \frac{X}{L} = \sum_{n=1}^{\infty} B'_n \sin \frac{n\pi x}{L} \tag{78}$$

and

$$B'_n = \frac{2}{L} \int_0^L \left[f(x) - T_0 \left(\frac{L-x}{L}\right) - T_1 \frac{X}{L} \right] \sin \frac{n\pi x}{L}\, dx \,. \tag{79}$$

Thus, after some rearrangement the complete solution of the original B. V. P. in eqs. (61), (62), (63), and (64) is

$$u(x,t) = T_0 \left(\frac{L-x}{L}\right) + T_1 \frac{x}{L} + \sum_{n=1}^{\infty} B'_n e^{-n^2\pi^2\kappa t/L^2} \sin \frac{n\pi x}{L} \tag{80}$$

where

$$B'_n = \frac{2}{L} \int_0^L \left[f(x) - T_0 \left(\frac{L-x}{L}\right) - T_1 \frac{X}{L} \right] \sin \frac{n\pi x}{L}\, dx \,. \tag{81}$$

Case 2: This problem corresponds to thermally insulated faces. The B. V. P. is then

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0, \quad (0 < x < L,\ 0 < t < \infty) \tag{82}$$

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \quad (0 \le t < \infty) \tag{83}$$

$$u(x,0) = f(x), \quad (0 \le xleqL) \,. \tag{84}$$

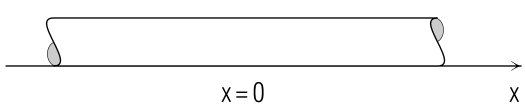
The solution can be obtained in the same way as in Case 1 except the presence of the time derivatives in eq. (83) results in a Fourier cosine series on $[0, L]$. Thus, the solution is

$$u(x,t) = \frac{1}{2} A'_0 \sum_{n=1}^{\infty} A'_n e^{-\dfrac{n^2\pi^2\kappa t}{L^2}} \cos \frac{n\pi x}{L} \tag{85}$$

where

$$A'_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 0, 1, 2, 3, \ldots \tag{86}$$

3f. Temperatures in an Infinite and Semi-infinite Bar. As a last application of Fourier analysis to the solution of P. D. E. , the Fourier integrals will be used to find temperatures in an infinite and semi-infinite bar. The B. V. P. for an infinite bar is as follows:



$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0, \quad (-\infty < x < \infty,\ 0 < t < \infty) \tag{87}$$

$$|u(x,t)| < M, \quad (-\infty < x < \infty,\ 0 < t < \infty) \tag{88}$$

and

$$u(x,0) = f(x), \quad (-\infty < x < \infty) \,. \tag{89}$$

Note: M is a constant, and $f(x)$ is known.

The bar extends to $\pm\infty$ in the x-direction, but it has negligible cross-sectional area so the temperature in the bar is a function of x alone (as well as time t). A solution to the P. D. E. can be written as

$$u_\omega(x,t) = u(\omega) e^{-\kappa\omega^2 t} e^{-\imath\omega x} \tag{90}$$

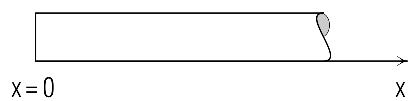
since $\partial u_\omega/\partial t = -\kappa\omega^2 u_\omega$ and $\partial^2 u_\omega/\partial x^2 = -\omega^2 u_\omega$. The proportionality constant can depend on ω. Thus, a complete solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\omega) e^{-\kappa\omega^2 t} e^{-\imath\omega x}\, d\omega \tag{91}$$

where

$$u(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{+\imath\omega x}\, dx\,. \tag{92}$$

The B. V. P. for a semi-infinite bar is as follows:



$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0, \;\; (0 < x < \infty,\; 0 < t < \infty) \tag{93}$$

$$|u(x,t)| < M, \;\; (0 < x < \infty,\; 0 < t < \infty) \tag{94}$$

$$u(0,t) = T_0, \;\; 0 < t < \infty) \tag{95}$$

$$u(x,0) = f(x), \;\; 0 < x < \infty)\,. \tag{96}$$

The B. C. at $x = 0$ corresponds to a constant temperature on the face of the bar at $x = 0$. The problem can be reformulated in terms of $v(x,t)$ where:

$$v(x,t) = u(x,t) - T_0\,. \tag{97}$$

Then, the new B. V. P. becomes

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \;\; (0 < x < \infty,\; 0 < t < \infty) \tag{98}$$

$$|v(x,t)| < M', \;\; (0 < x < \infty,\; 0 < t < \infty) \tag{99}$$

$$v(0,t) = 0, \;\; 0 < t < \infty) \tag{100}$$

$$v(x,0) = f(x) - T_0, \;\; 0 < x < \infty)\,. \tag{101}$$

A solution to the P. D. E. is simply

$$v_\omega(x,t) = v(\omega) e^{-\kappa\omega^2 t} \sin\omega x \tag{102}$$

since

$$\frac{\partial v_\omega}{\partial t} = -\kappa\omega^2 v_\omega \tag{103}$$

and

$$\frac{\partial^2 v_\omega}{\partial x^2} = -\omega^2 v_\omega\,. \tag{104}$$

Also

$$v_\omega(0,t) = 0\,. \tag{105}$$

The last B. C. can be fulfilled by integrating over ω from 0 to ∞. A factor of $\sqrt{2/\pi}$ is included. Thus,

$$v(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} v(\omega) e^{-\kappa\omega^2 t} \sin\omega x\, d\omega \tag{106}$$

where

$$v(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [f(x) - T_0] \sin\omega x\, dx\,. \tag{107}$$

Thus, the solution to the original B. V. P. is

$$u(x,t) = v(x,t) + T_0 \tag{108}$$

where $v(x,t)$ is given by eqs. (106) and (107) above.

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