PHYSNET

MISN-0-479

# DIFFERENTIAL VECTOR CALCULUS

# Math Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DIFFERENTIAL VECTOR CALCULUS

by

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Title: **Differential Vector Calculus**

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**Input Skills**:

1. Vocabulary: electrostatic potential field $\phi(\vec{r})$, Maxwell's equations, magnetic induction, electric field, free charge density, and current density.
2. Unknown: assume (MISN-0-478).

**Output Skills (Knowledge)**:

K1. Write down the definition or explanation of each of the following terms and concepts: del operator $\vec{\nabla}$, gradient (of a scalar field), geometric interpretation of gradient, divergence of a vector field, curl of a vector field, geometric meaning of divergence, Laplacian of scalar field, Laplace's equation $\nabla^2\phi = 0$.

K2. Verify simple vector identities involving the del operator, including successive applications of del.

**Output Skills (Rule Application)**:

R1. Calculate the gradient and Laplacian of a scalar field given the analytic form of the scalar field.

R2. Calculate the divergence and curl of a vector field given the analytic form of the vector field.

**External Resources (Required)**:

1. G. Arfken, Mathematical Methods for Physicists, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).



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New authors, reviewers and field testers are welcome.



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## 1. Introduction

The del operator $\vec{\nabla}$ which is introduced in this unit is a differential vector operator. Del must satisfy all the rules for vectors and partial differentiation. The concepts of the gradient of a scalar field and the divergence and curl of a vector field are central to all of theoretical physics from Newtonian mechanics to quantum field theory. The gradient of a scalar field has occurred already in your career in physics as an undergraduate. The electrostatic force $\vec{F}(\vec{r})$ on a particle of charge $q$ at position $r$ in an electrostatic potential field $\phi(\vec{r})$ is just given by:

$$\vec{F} = -q\vec{\nabla}\phi\,,$$

where $\vec{\nabla}\phi$ is "the gradient of $\phi$" or simply "del $\phi$." The operations of divergence and curl have been seen already also. For example, in vacuum with no charges or currents present in a region, the electric field $\vec{E}(\vec{r},t)$ and magnetic induction $\vec{B}(\vec{r},t)$ satisfy these equations:

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 0\\ \vec{\nabla}\times\vec{E} &= -\partial B/\partial t\\ \vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{B} &= \mu_0\epsilon_0\partial E/\partial t\,.\end{aligned}$$

Here $\vec{\nabla}\cdot\vec{E}$ is "the divergence of $E$" and $\vec{\nabla}\times\vec{E}$ is "the curl of $E$." In the introduction to MISN-0-478, we saw that temperature $T$ can be a scalar field $T(\vec{r},t)$ which also can depend on time. In a later unit, we shall see that the flow of heat is determined by $\vec{\nabla}T$. These are just a few instances where del, $\vec{\nabla}$, plays an important role in the mathematical analysis of physical phenomena.

## 2. Procedures

1. Read pages 125 - 127 of Spiegel. Begin with the section on "Limits, Continuity, etc." Do not include the section on "Orthogonal Curvilinear Coordinates" on page 127.





2. Read sections 1.6 through 1.9.
3. Read the Supplementary Notes on Maxwell's Equations. Read Example 1.9.2 on page 50 of Arfken.
4. Underline in the texts or write out the definitions and explanations of the terms and concepts of Output Skill K1.
5. Read through Solved Problems 5.26 to 5.29 in Spiegel in derivatives, gradient, divergence, and curl.
6. Solve these problems in Arfken:

| | |
|---|---|
| 1.6.2 (Gradient; normals) | 1.8.13 (Vector identity; curl) |
| 1.7.6 (Divergence) | 1 9.2 (Vector identity; successive curl) |
| 1.8.11 (Vector identity; curl) | 1.9.3 (Successive application of del) |

Note: Do *not* write out the vectors in component form when solving problems 1.8.11 and 1.8.13 of Arfken. Use the technique outlined in example 1.8.2 of Arfken and at the end of the Supplementary Notes.

## 3. Supplementary Notes

**Maxwell's Equations of Electromagnetic Theory**

(MKS Units-Vacuum)

Let $\vec{E}(\vec{r},t)$ be the electric field, $\vec{B}(\vec{r},t)$ be the magnetic induction, $\rho(\vec{r},t)$ represents the free charge density and $\vec{J}(\vec{r},t)$ represents the corresponding current. Then, let:

$$\vec{D} = \epsilon_0\vec{E}$$

and

$$\vec{B} = \mu_0\vec{H}\,.$$

Maxwell's equations are then:

$$\begin{aligned}\vec{\nabla}\cdot\vec{D} &= \rho; &\quad \vec{\nabla}\times\vec{E} &= -\partial\vec{B}/\partial t\\ \vec{\nabla}\cdot\vec{B} &= 0; &\quad \vec{\nabla}\times\vec{H} &= \partial\vec{D}/\partial t + \vec{J}\ .\end{aligned}$$

The magnetic induction can always be written in terms of a vector potential $\vec{A}(\vec{r})$ as

$$\vec{B} = \vec{\nabla}\times\vec{A}\,.$$

In the case of electrostatics and magnetostatics, when the various fields become independent of time, Maxwell's equations become

$$\vec\nabla\cdot\vec D=\rho;\qquad \vec\nabla\times\vec E=0$$
$$\vec\nabla\cdot\vec B=0;\qquad \vec\nabla\times\vec H=\vec J\ .$$

The electrostatic field can always be derived from a scalar potential $\phi(\vec r)$ as

$$\vec E=-\vec\nabla\phi(\vec r)\,.$$

It should be noted that problem 1.11.6 and 1.12.4 in Arfken gives boundary conditions on the displacement $D$ and magnetic intensity $H$, respectively, due to the presence of free charge and current at a boundary.

## Application of Del in Products

The discussion on page 46 of Arfken including example 1.8.2 is meant to illustrate a method which will shorten the calculation of vector identities in which del appears in a product. The idea is to use the regular BAC-CAB rule keeping in mind that del also must satisfy partial differentiation rules. As an example, problem 1.8.12 will be solved.

1.8.12 Show that

$$\vec\nabla(\vec A\cdot\vec B)=(\vec A\times\vec\nabla)\times\vec B+(\vec B\times\vec\nabla)\times\vec A+\vec A(\vec\nabla\cdot\vec B)+\vec B(\vec\nabla\cdot\vec A)\,.$$

Solution: Consider the quantity $(\vec A\times\vec\nabla_B)\times\vec B$ where the subscript $B$ has been attached to $\vec\nabla$ to show that it only differentiates $\vec B$. Applying the BAC-CAB rule by rewriting the triple cross product but rearranging the terms so $\vec\nabla_B$ always appears to the left of $\vec B$ in the final result gives

$$(\vec A\times\vec\nabla_B)\times\vec B=-\vec B\times(\vec A\times\vec\nabla_B)=-[\vec A(\vec\nabla_B\cdot\vec B)-\vec\nabla_B(\vec A\cdot\vec B)]\,.$$

In the same way,

$$(\vec B\times\vec\nabla_A)\times\vec A=-\vec A\times(\vec B\times\vec\nabla_A)=-[\vec B(\vec\nabla_A\cdot\vec A)-\vec\nabla_A(\vec A\cdot\vec B)]\,.$$

Adding gives

$$(\vec A\times\vec\nabla)\times\vec B+(\vec B\times\vec\nabla)\times\vec A=\vec\nabla_A(\vec A\cdot\vec B)+\vec\nabla_B(\vec A\cdot\vec B)-\vec A(\vec\nabla\cdot\vec B)-\vec B(\vec\nabla\cdot\vec A)$$

where the subscripts $A$ and $B$ have been dropped wherever no ambiguity exists concerning which vector is being differentiated by del. Thus

$$\vec\nabla_A(\vec A\cdot\vec B)+\vec\nabla_B(\vec A\cdot\vec B)=(\vec A\times\vec\nabla)\times\vec B+(\vec B\times\vec\nabla)\times\vec A+\vec A(\vec\nabla\cdot\vec B)+\vec B(\vec\nabla\cdot\vec A)\,.$$





But

$$\vec\nabla(\vec A\cdot\vec B)=\vec\nabla_A(\vec A\cdot\vec B)+\vec\nabla_B(\vec A\cdot\vec B)$$

where the del operator on the left operates on both $\vec A$ and $\vec B$. Thus,

$$\vec\nabla(\vec A\cdot\vec B)=(\vec A\times\vec\nabla)\times\vec B+(\vec B\times\vec\nabla)\times\vec A+\vec A(\vec\nabla\cdot\vec B)+\vec B(\vec\nabla\cdot\vec A)\,.$$

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