PHYSNET

MISN-0-47

LAPLACE TRANSFORM FOR THE DAMPED DRIVEN OSCILLATOR

$$\mathcal{L}\{F(t)\}$$

Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

LAPLACE TRANSFORM FOR THE DAMPED DRIVEN OSCILLATOR

by

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Title: **Laplace Transform for the Damped Driven Oscillator**

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Version: 2/6/2013 Evaluation: Stage 1

Length: 1 hr; 16 pages

Input Skills:

1. Write the differential equation of motion for the damped driven oscillator and distinguish between the transient and steady-state solutions (MISN-0-31).

Output Skills (Knowledge):

K1. Define the Laplace Transform of a function of a single variable and state the conditions under which the transform exists.

K2. Given the damping parameter λ, the driven function, and the natural frequency $\omega_0 = (k/m)^{1/2}$ of an oscillator, write down the expression for the resultant force (as a function of time) acting on the oscillator. Then, using Newton's Second Law, find the second order differential equation satisfied by the oscillator's displacement from equilibrium.

K3. Find the Laplace Transform of the damped, driven oscillator differential equation and then the Laplace Transform of the solution to this differential equation. Use the table of Laplace Transforms to find the sought-after solution. (This solution may be left in integral form).

Output Skills (Problem Solving):

S1. Evaluate the Laplace Transform of some given simple function (constant, square pulse, sum of integer powers of the variable, sine, cosine, and exponential).

S2. Calculate the Laplace Transforms of the derivatives of a function whose Laplace Transform is given.

External Resources (Required):

1. A Table of Laplace Transforms. For availability, see this module's *Local Guide*.

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1. Introduction

1a. A Useful Method for Solving Differential Equations. The Laplace Transform method for solving ordinary differential equations is especially useful for finding the solutions to the differential equations governing the simple harmonic oscillator, the damped harmonic oscillator, and the damped driven oscillator.[1] This method reduces the differential equation to an algebraic equation. The inverse Laplace Transform of the solution to this algebraic equation is then the solution to the differential equation. Many engineers find this technique especially attractive for solving the differential equations that they encounter.

2. Elementary Properties of the Laplace Transform

2a. Definition of the Laplace Transform. If $F(t)$ is specified for values of $t > 0$, then the Laplace Transform of $F(t)$ is defined by:

$$\text{Laplace Transform of } F(t) \text{ for a value of } s \equiv \int_0^\infty e^{-st} F(t)\, dt$$

We write this as:

$$\mathcal{L}\{F(t)\} \equiv \int_0^\infty e^{-st} F(t)\, dt.$$

The right side is a function of s, call it $f(s)$. The Transform exists for those values of s for which the integral converges.

2b. Laplace Transform of the Constant Function. Consider the simplest $F(t)$, a constant, call it A. Evaluating the transform integral:

$$f(s) = A \int_0^\infty e^{-st} dt = \frac{A}{s} \left[-e^{-st}\right]_{t=0}^{t=\infty} .$$

[1]These are phenomena for which Fourier Transforms, for example, do not exist.

For $s < 0$ and for $s = 0$, this integral doesn't converge (for negative s the upper limit gets infinite) so $f(s)$ exists only for values of $s > 0$, and

$$f(s) = \frac{A}{s}, \qquad (s > 0) .$$

2c. Laplace Transform of the Exponential Function. Another example is $F(t) = e^{at}$, where a is some constant:

$$f(s) = \int_0^\infty e^{(a-s)t} dt.$$

If $a > s$ the exponent is always positive and blows up as t gets infinite. However, for $s > a$ the integrand approaches zero for large t. The integral is:

$$\frac{1}{(a-s)} e^{(a-s)t}\Big|_{t=0}^{t=\infty} .$$

It is zero at the upper limit if $s > a$ and finite at the lower limit (unless $s = a$). Hence,

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad (s > a).$$

2d. Relating Sine and Cosine to Exponential. Sine/cosine transforms can be related to exponential ones by:[2]

$$e^{\pm iat} = \cos at \pm i \sin at \,,$$

where $i = \sqrt{-1}$. Adding and subtracting these relations gives you two useful relations for the sine and cosine:

$$\sin at = \frac{1}{2i}(e^{iat} - e^{-iat}) \,,$$

$$\cos at = \frac{1}{2}(e^{iat} + e^{-iat}).$$

2e. Exercises with Simple Functions. Using the above results, try the following exercises.

▷ Find $\mathcal{L}\{F(t)\}$ for each of these functions:

(i) $\sin at$

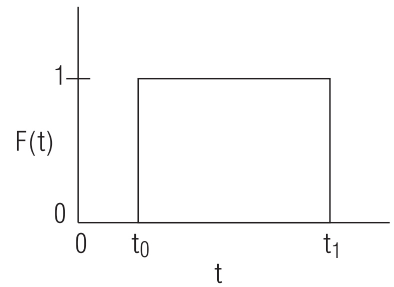
[2]See "Some Simple Functions in the Complex Plane" (MISN-0-59).

(ii) $\cos at$

(iii) t

(iv) t^2

$$\text{(v)}\quad \left.\begin{array}{rcl} F(t) & = & 0 \text{ if } t < t_0 \\ F(t) & = & 1 \text{ if } t_0 < t < t_1 \\ F(t) & = & 0 \text{ if } t > t_1 \end{array}\right\}$$



The answers you get should be :

(i) $\dfrac{a}{s^2 + a^2}$, for $s > 0$

(ii) $\dfrac{s}{s^2 + a^2}$, for $s > 0$

(iii) $\dfrac{1}{s^2}$, for $s > 0$

(iv) $\dfrac{2}{s^3}$, for $s > 0$

Note: by induction, you can show that: $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ for positive integer n.]

(v) $\dfrac{e^{-t_0 s}}{s}\left[1 - e^{-(t_1 - t_0)s}\right]$

2f. Laplace Transform of Derivatives. What about the Laplace Transform of $dF(t)/dt$ and higher derivatives? You can, of course, evaluate these directly if you know $F(t)$. However, the Laplace Transform of the derivatives of $F(t)$ can be expressed in terms of the Laplace Transform of $F(t)$. Let's look at the first derivative:

$$\mathcal{L}\{\frac{dF(t)}{dt}\} = \int_0^\infty e^{-st}\frac{dF}{dt}dt\ .$$

Using:

$$\frac{d}{dt}[e^{-st}F] = F\frac{de^{-st}}{dt} + e^{-st}\frac{dF}{dt},$$

the integral may be written:

$$\int_0^\infty \frac{d}{dt}[e^{-st}F]dt - \int_0^\infty \frac{d[e^{-st}]}{dt}F dt\,.$$

The first integral is just $e^{-st}F(t)$, the second is:

$$s\int_0^\infty e^{-st}F(t)dt,$$

which is: $s\,\mathcal{L}\{F(t)\}$. Hence:

$$\begin{aligned} \mathcal{L}\left\{\frac{dF(t)}{dt}\right\} &= e^{-st}F(t)\big|_{t=\infty} - e^{-st}F(t)\big|_{t=0} + s\mathcal{L}\{F(t)\} \\ &= sf(s) - F(0)\,, \end{aligned}$$

where $F(0)$ is the function $F(t)$ evaluated at $t = 0$. Try this yourself for the second derivative and show that:

$$\mathcal{L}\left\{\frac{d^2F(t)}{dt^2}\right\} = s^2 f(s) - sF(0) - F'(0)\,,$$

where:

$$F'(0) \equiv \left.\frac{dF(t)}{dt}\right|_{t=0}$$

is the first derivative of $F(t)$ evaluated at $t = 0$.

3. Solution for Damped Driven Oscillator

3a. Transform of Second Order Linear Differential Equation. Now consider the second order differential equation:

$$mx''(t) + \lambda x'(t) + kx(t) = F(t),$$

where:

$$x' \equiv \frac{dx(t)}{dt}, \qquad \text{and} \qquad x'' \equiv \frac{d^2x(t)}{dt^2}\ .$$

You'll recognize this as the differential equation of motion for the damped driven oscillator. For example, a mass m at the end of a spring constant k with damping coefficient λ which is being driven by an externally applied force $F(t)$. At any instant of time t, the net force on the mass m is

$F(t) - \lambda dx/dt - kx$.[3] The quantities $x''(t)$ and $x'(t)$ are the first and second derivative of $x(t)$, the displacement from equilibrium with respect to the independent variable t,the time. If we divide by m our equation becomes:

$$x'' + 2\gamma x' + \omega^2 x = G(t)\,.$$

$$\text{Here:}\qquad 2\gamma \equiv \frac{\lambda}{m},\qquad \omega^2 \equiv \frac{k}{m},\qquad G(t) \equiv \frac{F(t)}{m}$$

Taking the Laplace Transform of both sides yields:

$$\mathcal{L}\{x''\} + 2\gamma\mathcal{L}\{x'\} + \omega^2\mathcal{L}\{x\} = \mathcal{L}\{G(t)\}.$$

Using the notation:

$$f(s) \equiv \mathcal{L}\{x(t)\}\,,\qquad \text{and:}\qquad g(s) \equiv \mathcal{L}\{G(t)\},$$

and using the above developed relations for the Laplace Transforms of the first and second derivatives, we arrive at the equation:

$$s^2 f(s) - s\,x(0) - x'(0) + 2\gamma s f(s) - 2\gamma x(0) + \omega^2 f(s) = g(s).$$

This is just an algebraic equation that must be satisfied by $f(s)$, given $g(s)$ and the constants $x(0)$ and $x'(0)$. Solving for $f(s)$ we obtain:

$$f(s) = \frac{(s+2\gamma)x(0) + x'(0) + g(s)}{s^2 + 2\gamma s + \omega^2}\,.$$

3b. Inverse Laplace Transform. What we wanted to find was $x(t)$ for all values of t. Instead what we have is $f(s)$, the Laplace Transform of $x(t)$. Knowing $f(s)$, to find $x(t)$ you need to evaluate the inverse Laplace Transform of $f(s)$:

$$x(t) \equiv \mathcal{L}^{-1}\{f(s)\}\,,\qquad \text{where:}\qquad f(s) = \mathcal{L}\{x(t)\}\,.$$

The standard procedure for finding the inverse Laplace Transform is to use a table which displays the Laplace Transform of a large variety of functions (and hence the inverse transform as well.) For example, suppose $\lambda = 0$ and $F(t) = 0$. Then our equation is the equation of motion for the simple harmonic oscillator:

$$x''(t) + \omega_0^2 x(t) = 0\,,$$

[3]See "Damped Mechanical Oscillations" (MISN-0-29), "Damped Driven Oscillations" (MISN-0-31), and "Computer Algorithm for the Damped Driven Oscillator"(MISN-0-39) for more on the damped driven oscillator.

and taking the Laplace Transform of this equation and solving it for the Laplace Transform of $x(t)$, we have:

$$\mathcal{L}\{x(t)\} \equiv f_0(s) = \frac{s\,x(0) + x'(0)}{s^2 + \omega_0^2}\,.$$

But we saw previously that:

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2},\qquad \text{and:}\qquad \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}\,,$$

so our transform is:

$$f(s) = x(0)\left[\frac{s}{s^2 + \omega_0^2}\right] + \frac{x'(0)}{\omega_0}\left[\frac{\omega_0}{s^2 + \omega_0^2}\right].$$

Then the function whose transform this is, must be:

$$x(t) = x_0 \cos\omega_0 t + \frac{x'(0)}{\omega_0}\sin\omega_0 t\,,$$

which you recognize as the solution to the differential equation that we started with, $x''(t) + \omega_0^2 x(t) = 0$. For the damped driven oscillator, the solution to the problem is the inverse transform of the $f(s)$ we previously found:

$$\begin{aligned} x(t) \qquad &= \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{(s+2\gamma)x(0) + x'(0) + g(s)}{s^2 + 2\gamma s + \omega_0^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s+2\gamma)x(0) + x'(0)}{s^2 + 2\gamma s + \omega_0^2}\right\} + \mathcal{L}^{-1}\left\{\frac{g(s)}{s^2 + 2\gamma s + w_0^2}\right\}, \end{aligned}$$

using the linearity of the inverse Laplace Transform operation. The second transform arises only if there is a driving force. It is the "transient."[4]

3c. Transient Solution. Looking at just the transient term for the time being (this is the solution to the damped oscillator problem[5]), and scanning a table of Transforms we see that:

$$\mathcal{L}\left\{e^{-at}\sin\omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2}\,,$$

and:

$$\mathcal{L}\left\{e^{-at}\cos\omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

[4]See "Damped Driven Oscillations; Mechanical Resonances"(MISN-0-31).
[5]See "Damped Mechanical Oscillations" (MISN-0-29).

So we know, for example, the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin\omega t\,.$$

Our transient term looks to be almost of this form. If we complete the square in the denominator,

$$s^2+2\gamma s+\omega_0^2 = (s+\gamma)^2+\omega_0^2-\gamma^2\,,$$

then:

$$x_{\text{transient}}(t) = \mathcal{L}^{-1}\left[\frac{(s+\gamma)x(0)+\gamma x(0)+x'(0)}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right]\,,$$

$$x_{\text{transient}}(t) = \mathcal{L}^{-1}\left[\frac{(s+\gamma)x(0)}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right] + \mathcal{L}^{-1}\left[\frac{\gamma x(0)+x'(0)}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right]\,,$$

$$\mathcal{L}^{-1}\left\{\frac{(s+\gamma)x(0)}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right\} = x(0)e^{-\gamma t}\cos\sqrt{\omega_0^2-\gamma^2}\,t\,,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right\} = \frac{1}{\sqrt{\omega_0^2-\gamma^2}}e^{-\gamma t}\sin\sqrt{\omega_0^2-\gamma^2}\,t.$$

Then the full transient solution is:

$$x_{\text{tran}}(t) = x(0)e^{-\gamma t}\cos\omega t + \frac{\gamma x(0)+x'(0)}{\omega}e^{-\gamma t}\sin\omega t\,,$$

where: $\omega \equiv \sqrt{\omega_0^2-\gamma^2}$. If you put in $t=0$, the right side is only $x(0)$. Taking the first derivative and setting $t=0$, you again get only $x'(0)$ on the right side. This again shows that the constants $x(0)$ and $x'(0)$ are the initial values of the displacement and the velocity. As $t\to\infty$, this transient term dies out because of the exponential.

3d. Steady-State Solution. What about the steady-state solution, the one that comes about because of the driving force? That solution is:

$$x_{\text{steady}}(t) = \mathcal{L}^{-1}\left\{\frac{g(s)}{(s+\gamma)^2+\omega_0^2-\gamma^2}\right\}\,,$$

and depends of course on the specific time dependence of the driving force [and hence the functional form of $g(s)$].

We now use the valuable Convolution Property of Laplace Transforms, which states:

If: $\quad \mathcal{L}^{-1}\{g(s)\} = G(t)\,,\qquad$ and: $\quad \mathcal{L}^{-1}\{f(s)\} = F(t),$

then: $\quad \mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(t-u)G(u)du.$

For our steady-state solution:

$$x_{\text{steady}}(t) = \frac{1}{\sqrt{\omega_0^2-\gamma^2}}\int_0^t e^{-\gamma(t-u)}\sin[(\omega_0^2-\gamma^2)^{1/2}(t-u)]\;G(u)\;du\,.$$

This is a messy integral even for the simplest case of a constant driving force. Let's look at a simplification, a case of no damping, $\gamma=0$ and a sinusoidal driving force:

$$G(t) = \frac{F(t)}{m} = \frac{F_0}{m}\cos\omega t\,,$$

$$x_{\text{steady}}(t) = \frac{F_0}{m\omega_0}\int_0^t \sin[\omega_0(t-u)]\cos\omega u\,du.$$

Using: $\sin\,(\omega_0 t-\omega_0 u) = \sin\omega_0 t\;\cos\omega_0 u - \cos\omega_0 t\;\sin\omega_0 u$

we find:

$$\begin{aligned} x_{\text{steady}}(t) &= \frac{F_0}{m\omega_0}[\sin\omega_0 t\int_0^t \cos\omega_0 u\;\sin\omega u\;du - \\ &\quad \cos\omega_0 t\int_0^t \sin\omega_0 u\;cos\omega u\;du] \\ &= \frac{F_0}{m\omega_0}\left[\sin\omega_0 t\left(\frac{\sin(\omega_0-\omega)t}{2(\omega_0-\omega)}+\frac{\sin(\omega_0+\omega)t}{2(\omega_0+\omega)}\right)+\right. \\ &\quad \left.\cos\omega_0 t\left(\frac{\cos(\omega_0-\omega)t}{2(\omega_0-\omega)}+\frac{\cos(\omega_0+\omega)t}{2(\omega_0+\omega)}-2\right)\right]. \end{aligned}$$

Notice that when the driving frequency, ω, is close to the natural frequency of the oscillator, ω_0, x gets very large. The system is near resonance.[6]

Acknowledgments

We thank Andres Ordonez for pointing out an error, now fixed. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

[6] See "Damped Driven Oscillations; Mechanical Resonances" (MISN-0-31).

LOCAL GUIDE

The book *Theory and Problems of Laplace Transforms* has been placed on on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building.

PROBLEM SUPPLEMENT

Note: Problems 13-14 also occur in this module's *Model Exam.*

1. How is the Laplace Transform of $F(t)$ defined? [K]
2. For what values of s is it defined? [E]
3. Evaluate directly the Laplace Transform, $f(s)$, of $F(t) = 7t + 8$. [I]
4. For what values of s is the above $f(s)$ defined? [A]
5. Without referring back to the text, evaluate the Laplace Transform of $d^2F(t)/dt^2$. Your answer should be expressed in terms of the Laplace Transform of $F(t)$ and some constants. [G]
6. What are the constants in your answer to 5? [J]
7. Consider the differential equation $5F''(t) - 6F'(t) + F(t) = 10$, subject to these boundary conditions at $t = 0$: $F(0) = 0$ and $F'(0) = -2$. What algebraic equation must the Laplace Transform of $F(t)$ satisfy? [B] Solve this equation. [H]
8. Find the inverse Laplace Transform to the $f(s)$ you found in the above problem. Use Appendix A (entry 11) and Appendix B (entry 16) on pages 245 and 246 of Schaum or some other reference book's entries. [C]
9. A damped harmonic oscillator has a damping term proportional to the velocity of the oscillator mass with a damping constant equal to 4.0 N s/m. The mass of the oscillator is 5.0 kg while the spring constant is 2.0×10^1 N/m. Determine the resultant force on the oscillator mass as a function of time. [L]
10. What is the differential equation of motion for this oscillator? [D]
11. If the oscillator is at rest at $t = 0$, but displaced by amount $x = +0.1$ m from equilibrium, find the algebraic equation satisfied by the Laplace Transform of $x(t)$. [F]. Solve this equation. [N]
12. Find the inverse Laplace-transform to this, and hence the solution to the problem. Don't just plug into a form you find for the solution. Evaluate it step-by-step as in the text. [M]

13. Evaluate the Laplace Transform of:
    $F(t) = at^2 + bt$. [P]
14. Determine the formal expression for
    $\mathcal{L}\{\frac{d^3F(t)}{dt^3}\}$,
    in terms of $f(s) \equiv \mathcal{L}\{F(t)\}$ and specified $t = 0$ constants. [O]

Brief Answers:

A. for all $s > 0$.

B. $5s^2 f + 10 - 6sf + f = 10$/s.

C. $F = 10(1 - e^{t/5})$.

D. $x'' + 0.8x' + 4x = 0$.

E. For those values of s for which the integral converges.

F. $s^2 f - 0.1s + 0.8sf - 0.08 + 4f = 0$.

G. $s^2 f(s) - sF(0) - F'(0)$.

H. $f = \frac{10}{s(5s-1)(s-1)} - \frac{10}{(5s-1)(s-1)}$.

I. $\frac{8s+7}{s^2}$.

J. The function F and its first derivative evaluated at $t = 0$.

K. $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} f(t)\, dt$.

L. $-4v(t) - 20x(t)$

M. $x(t) = exp(-0.4t)\{0.1\cos(1.96t) + 0.02\sin(l.96t)\}$

N. $\frac{0.1s + 0.08}{s^2 + 0.8s + 4}$.

O. $s^3 f(s) - s^2 f(0) - sf'(0) - f''(0)$.

P. $(2a + bs)/s^3$.

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*. One or more of these skills, or none, may be on the actual exam.
2. Evaluate the Laplace Transform of:
   $F(t) = at^2 + bt$.
3. Determine the formal expression for
   $\mathcal{L}\{\frac{d^3F(t)}{dt^3}\}$,
   in terms of $f(s) \equiv \mathcal{L}\{F(t)\}$ and specified constants.

Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 13.
3. See this module's *Problem Supplement*, problem 14.