PHYSNET

MISN-0-469

FOUR-TENSORS AND THE LORENTZ GROUP

Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

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by

C. P. Frahm

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Title: **Four-Tensors and the Lorentz Group**

Author: C. P. Frahm, Dept. of Physics, Illinois State Univ.

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Input Skills:

1. Vocabulary: matrix, transpose, inverse, matrix multiplication.
2. State the Lorentz transformation in matrix form for a boost along a coordinate axis.
3. State the definition of the invariant spacetime interval.
4. Unknown: assume (MISN-0-468).

Output Skills (Knowledge):

K1. Define general Lorentz transformations in terms of the invariance of the squared interval. Express general Lorentz transformations using index notation (including the summation convention) and matrices. Show that the transformation matrix must satisfy: $a^T ga = g$; $a^{-1} = ga^T g$.

K2. Concerning four-tensors: (a) Define four-tensors in terms of the transformation properties of their components using contravariant or covariant indices. (b) Demonstrate the use of the metric tensor in raising or lowering indices. (c) Construct new tensors from given tensors by addition, multiplication, contraction and differentiation. (d) Show that the four-gradient or "box" operator behaves as a covariant vector while the D'Alembertian behaves as a scalar.

K3. Define the velocity and acceleration four-vectors. Use their transformation properties to establish the transformation properties of the corresponding three-vectors.

External Resources (Required):

1. W. Rindler, *Essential Relativity* (Van Nostrand, 1977).

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1. Introduction

For many of the topics to be covered in the remainder of this course, it is convenient and even necessary to utilize the power of tensor analysis. This unit constitutes an introduction to tensors associated with Lorentz transformations. A subsequent unit will deal in more detail with general tensor analysis. This is also a convenient time to introduce some efficient notation and to explore some of the more subtle aspects of Lorentz transformations.

2. Procedures

1a-b. In MISN-0-466 the special Lorentz transformations were given in matrix form

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

It is common practice to make the identification

$$(x^0, x^1, x^2, x^3) = (t, x, y, z)$$

where the superscripts are indices *not* exponents. Then the general linear transformation can be written in the form

$$x^{\mu'} = \sum_{\nu=0}^{3} a^{\mu}_{\nu} x^{\nu}$$

where the a^{μ}_{ν} are the elements of the transformation matrix. It is convenient to use the Einstein summation convention wherein repeated indices - one upper and one lower - are understood to be summed from zero to three. The general linear transformation is then written

$$x^{\mu'} = a^{\mu}_{\nu} x^{\nu}$$

This equation can also be conveniently written in matrix form

$$x' = ax$$

where x and x' are column matrices and a is a square matrix. Note: the horizontal positioning of the indices on a^{μ}_{ν} has been chosen to facilitate the change to and from the matrix form - left for row and right for column. The vertical positioning will be discussed shortly.

Not all linear transformations are Lorentz transformations. However, any linear transformation that leaves the squared interval invariant is a Lorentz transformation.

▷ Exercise - Use this definition of the general Lorentz transformation to show that all Lorentz transformations leave the speed of light invariant.

To see what conditions this definition imposes on the transformation matrix consider two events with space-time coordinates x^{μ}_1, and x^{μ}_2. The square interval between these two events can be written in the equivalent forms (in frame S):

$$\begin{aligned}(\Delta S)^2 &= (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= (x^0_2 - x^0_1)^2 - (x^1_2 - x^1_1)^2 - (x^2_2 - x^2_1)^2 - (x^3_2 - x^3_1)^2 \\ &= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2\end{aligned}$$

$$= (\Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

$$= (\Delta x)^T g \Delta x$$

$$= \Delta x^{\mu} g_{\mu\nu} \Delta x^{\nu}$$

where g is the metric matrix with diagonal elements (1, −1, −1, −1).

In another inertial frame S' the squared interval between the same two events would be given by

$$(\Delta S')^2 = (\Delta x')^T g \Delta x'$$

with $\Delta x'$ related to Δx by a linear transformation

$$\Delta x' = a \Delta x$$

Thus

$$(\Delta S')^2 = (a\Delta x)^T g(a\Delta x)$$
$$= (\Delta x)^T a^T g a \Delta x$$

Now if the transformation is to be a Lorentz transformation according to the definition given earlier, the squared interval must be invariant. Thus

$$(\Delta S')^2 = (\Delta S)^2$$

This requires

$$(\Delta x)^T a^T g a \Delta x = \Delta x^T g \Delta x$$

Since Δx is arbitrary, it must be concluded that

$$a^T g a = g$$

A transformation matrix is a Lorentz transformation if and only if it satisfies this relation.

▷ Exercise - Show from the relation

$$a^T g a = g$$

that all Lorentz transformations satisfy

$$(\det a)^2 = 1$$

From this exercise it is clear that for a Lorentz transformation a must be non-singular and hence possesses an inverse (a^{-1}).

▷ Exercise - Show that

$$a^{-1} = g a^T g$$

In index form

$$x^{\mu'} = a^{\mu}_{\nu} x^{\nu}$$

and

$$x^{\mu} = (a^{-1})^{\mu}_{\nu} x^{\nu'}$$

are inverse relations. Thus

$$(a^{-1})^{\mu}_{\nu} a^{\nu}_{\rho} = \delta^{\mu}_{\rho}$$

where δ^{μ}_{ρ} is the Kronecker delta (giving the elements of the 4×4 unit matrix).

c. A set of quantities is said to constitute a group if it possesses a composition rule (called multiplication) which satisfies these properties:

(1) closure - the product of any two elements in the set is an element of the set.

(2) associativity - for any three elements of the set

$$(a_1 a_2) a_3 = a_1 (a_2 a_3)$$

(3) identity - the set includes an identity element (e or I or 1) such that for any element in the set

$$aI = Ia = a$$

(4) inverse - for every element a in the set there exists another element a^{-1} in the set such that

$$a a^{-1} = a^{-1} a = I$$

▷ Exercise - Show that the set of all 4×4 matrices that satisfy

$$a^T g a = g$$

constitutes a group (the Lorentz group).

d. (Optional) The squared interval is invariant under an even larger set of transformations than the Lorentz group - the Poincaŕe transformations (inhomogeneous Lorentz transformations).

$$x^{\mu'} = a^{\mu}_{\nu} x^{\nu} + B^{\mu}, \text{ with } a^T g a = g$$

Note that the inhomogeneous part constitutes a shift of the space-time origin. Each Poincaŕe transformation can be represented by the ordered pair (a, B).

▷ Exercise - Consider two consecutive Poincaŕe transformations ($S \to S' \to S''$) and show that the composition rule for Poincaŕe transformations is

$$(a', B')(a, b) = (a'a, a'B + B')$$

▷ Exercise - Show, using the above composition rule, that the set of all Poincaŕe transformations constitutes a group - the Poincaŕe group.

2. a. Read section 4.4 of Rindler. Equation 4.24 in Rindler illustrates the transformation rule for contravariant indices. In general the numbers

$$A^{\mu\nu\ldots}_{\rho\sigma\ldots}$$

are the components of a (Lorentz or four) tensor, contravariant in the superscripted indices, $\mu\nu\ldots$ and covariant in the subscripted indices $\rho\sigma\ldots$, if under a Lorentz transformation

$$x^{\mu'} = a^{\mu}_{\nu} x_{\nu}$$

they transform according to

$$A^{\mu\nu\ldots'}_{\rho\sigma\ldots} = a^{\mu}_{\lambda}\, a^{\nu}_{\omega} \ldots A^{\lambda\omega\ldots}_{\eta\tau\ldots}\, (a^{-1})^{\eta}_{\rho}\, (a^{-1})^{\tau}_{\sigma} \ldots$$

The total number of indices on A is the rank of the tensor. A scalar is a zero rank tensor while a vector is a first rank tensor.

Note: The order of factors on the right-hand side is immaterial and has been chosen for convenience. However, the location of indices is crucial and should be studied carefully.

Note: It is sometimes convenient to represent a second rank tensor by a matrix. However, one should not mentally equate tensors with matrices. A matrix is simply a rectangular array of numbers endowed with a multiplication rule while tensor components have definite transformation properties under coordinate transformations. Tensors of rank greater than 2 cannot be represented by matrices.

Comment: A tensor is a geometrical object just as a vector is a geometrical object (usually represented by an arrow). It is an invariant or absolute quantity independent of the coordinate system just as a vector has an absolute meaning (direction and magnitude) independent of the coordinate system. Only the components of the tensor change from coordinate system to coordinate system.

▷ Exercise - Consider the following equation

$$A^{\mu\nu}_{\rho} = B^{\mu\lambda} C^{\omega}_{\lambda\rho} D^{\nu}_{\omega}$$

where A, B, C and D are Lorentz tensors of the indicated rank. Show that such a tensor equation retains its form under an arbitrary Lorentz transformation.

▷ Exercise - If the metric $g_{\mu\nu}$ is treated as a second rank tensor, show that it is numerically invariant under arbitrary Lorentz transformations.

b. It is often convenient to use the metric tensor to lower indices

$$A^{\mu}_{\nu} \equiv A^{\mu\rho} g_{\rho\nu}$$

Note: Since $A^{\mu\rho}$ and $g_{\rho\nu}$ are both tensors A^{μ}_{ν} is also a tensor (See above exercise). In fact A^{μ}_{ν} and $A^{\mu\rho}$ may be thought of as different kinds of components of the same tensor.

Similarly, a contravariant form of the metric tensor

$$g^{\mu\nu} \equiv g_{\mu\nu}$$

can be used to raise indices.

▷ Exercise - Show that this is consistent by verifying that

$$g_{\mu\rho} g^{\rho\omega} g_{\omega\nu} = g_{\mu\nu}$$

In raising and lowering indices, one has to be mindful of the position of the indices since

$$A^{\sigma}_{\mu} = A_{\mu\rho} g^{\rho\sigma} \neq A_{\rho\mu} g^{\rho\sigma} = A_{\mu}^{\sigma}$$

▷ Exercise - Consider the space-time position of an event $x^{\mu} = (t, x, y, z)$. What are the covariant coordinates (components) of the event? (Note: x^{μ} is not a vector under Poincaŕe transformations).

▷ Exercise - The electromagnetic field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_x \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

where the rows (columns) are numbered from top to bottom (left to right) 0, 1, 2, 3 and the three vectors $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. (Express answers in matrix form).

a. Determine the contravariant components $F^{\mu\nu}$.

b. Determine the mixed components F^{μ}_{ν}.

c. Determine the mixed components F_{μ}^{ν} (Compare with (b)).

c. The operation of summing on one upper and one lower index is called contraction. The first exercise in procedure 2a established that this leads to a new tensor.

▷ Exercise - Show that $A^{\mu}B_{\mu}$ is a scalar if A and B are vectors.

New tensors can also be obtained from two given tensors by component addition provided the tensors are of the same rank and the same type components (covariant, contravariant or mixed) are used.

$$T^{\mu\nu}_{\rho\omega\eta} = R^{\mu\nu}_{\rho\omega\eta} + S^{\mu\nu}_{\rho\omega\eta}$$

▷ Exercise - Prove this assertion.

Multiplication of tensors always leads to a new tensor of higher rank (unless one is a scalar)

$$T^{\mu\nu}_{\rho\omega\eta} = R^{\mu\nu}_{\rho} S_{\omega\eta}$$

▷ Exercise - Prove this assertion

Differentiation of a tensor with respect to a scalar always yields a tensor of the same rank. (This is true for Lorentz tensors but is not valid for more general tensors as will be seen later in the course.)

d. The natural extension of the gradient or $\vec{\nabla}$ operator in three dimensions is the four-gradient operator which is conveniently represented by the notation

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

▷ Exercise - Show that ∂_μ transforms like the covariant components of a Lorentz vector.

The generalization of the Laplacian ∇^2 is the D'Alembertian or box operator

$$\Box \equiv -g^{\mu\nu}\partial_\mu\partial_\nu$$

▷ Exercise - Show that

$$\Box = \nabla^2 - \frac{\partial^2}{\partial t^2}$$

▷ Exercise - Show that the D'Alembertian is a Lorentz scalar operator.

Read Rindler, sections 4.1 - 4.3, to pick up a few loose ends.

3. The four-velocity and four-acceleration are defined in section 4.3 of Rindler, equations 4.8 and 4.11. In index notation these would read:

$$U^\mu = \frac{dx^\mu}{d\tau}, \quad A^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$

Special attention should be given to equation 4.10 and 4.12.

Comment: My preference is to list the zero-th component first so that

$$U^\mu = \gamma_u(1, \vec{u}) \text{ with } \gamma_u = (1-u^2)^{-1/2}$$

and

$$A^\mu = \gamma_u \frac{d}{dt}(\gamma_u, \gamma_u\vec{u})$$

Note that γ_u has nothing to do with transformations.

▷ Exercise - Show that

$$A^\mu = \gamma_u(\dot{\gamma_u}, \dot{\gamma_u}\vec{u} + \gamma_u\vec{a})$$

where the dots denote derivatives with respect to t.

▷ Exercise - Use the transformation law for the 4-velocity to show that for a boost at speed v in the $+x$ - direction

a. $\gamma'_u = \gamma\gamma_u(1 - vu_x)$

b.
$$u'_x = \frac{u_x - v}{1 - vu_x}, \quad u'_y = \frac{u_y}{\gamma(1 - vu_x)}$$

where $\gamma = (1-v^2)^{-1/2}$ (for the transformation)
and $\gamma_u = (1-u^2)^{-1/2}$, $\gamma'_u = (1-u'^2)^{-1/2}$

▷ Exercise - Use the transformation law for the 4 - acceleration to show that for a boost at speed v in the $+x$ - direction

a. $\gamma'_u\dot{\gamma'_u} = \gamma\gamma_u[(1 - vu_x)\dot{\gamma_u} - v\gamma_u a_x]$, and

b. the 3 - acceleration transforms in the way given by the exercise in MISN-0-468.

Read section 4.6 of Rindler. Note the timelike character of U^μ and the spacelike character of A^μ.

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