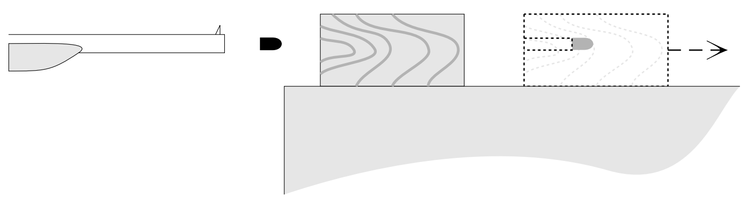
MISN-0-413

MOTION OF A SYSTEM OF PARTICLES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MOTION OF A SYSTEM OF PARTICLES

by

F. Reif, G. Brackett and J. Larkin

CONTENTS

Title: **Motion of a System of Particles**

Author: F. Reif, G. Brackett, and J. Larkin, Department of Physics, University of California, Berkeley.

Version: 4/30/2002 Evaluation: Stage 0

Length: 1 hr; 48 pages

Input Skills:

1. State the equation of motion for a particle (MISN-0-408).

Output Skills (Knowledge):

K1. Vocabulary: state of a system, total external force, mass of a system, center of mass, momentum.

K2. State how the principles of mechanics can be applied to predict the motion of systems of particles.

K3. State the equation of motion for a system of particles.

K4. State two properties of the mass of a system.

K5. State the principle of conservation of momentum.

Output Skills (Problem Solving):

S1. For a system of particles, solve problems using:

S2. Given the forces acting on the particles in a system, calculate the total internal and external forces acting on the system.

S3. Use the external forces on a system of particles to calculate the acceleration of its center of mass.

S4. Locate the position of the center of mass for systems with symmetry.

S5. Given the masses and velocities of the particles in a system at some time, apply the principle of conservation of momentum to calculate the momentum of the system at any time.

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

PROJECT STAFF

Andrew Schnepp	Webmaster
Eugene Kales	Graphics
Peter Signell	Project Director

ADVISORY COMMITTEE

D. Alan Bromley	Yale University
E. Leonard Jossem	The Ohio State University
A. A. Strassenburg	S. U. N. Y., Stony Brook

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

© 2002, Peter Signell for Project PHYSNET, Physics-Astronomy Bldg., Mich. State Univ., E. Lansing, MI 48824; (517) 355-3784. For our liberal use policies see:

`http://www.physnet.org/home/modules/license.html`.

MISN-0-413

MOTION OF A SYSTEM OF PARTICLES

Abstract:
In the preceding units we have studied the motion of individual particles and some of the important interactions between them. In the next few units we shall explore some of the general implications of our basic theoretical principles and shall thus develop methods useful for dealing with many important practical applications. In particular, we shall devote the present unit to extend our understanding of the motion of single particles in order to deal with systems consisting of many particles. (Unless explicitly stated otherwise, we shall again describe all motions relative to some convenient *inertial* frame.)

SECT.

A PREDICTIVE POWER OF MECHANICS

Our study of "mechanics" (the science of motion) has been based upon these fundamental principles: (1) A theory of motion summarized by the equation of motion $m\vec{a} = \vec{F}$. (2) Force laws which specify how the interaction between particles depends on their properties and relative positions. To assess the practical utility of these principles, we now ask this question: How effective are these principles for predicting the motion of a system consisting of many particles?

Consider an isolated system of N particles having known properties (e.g., known masses and charges) and interacting with each other by forces described by known force laws. (For example, consider the solar system consisting of the sun, the earth, and the other planets which interact by gravitational forces.) Suppose that we know the position vector $\vec{r}_0$ and velocity $\vec{v}_0$ of every particle at some instant of time t_0. Using the equation of motion $m\vec{a} = \vec{F}$, what predictions can we then make about the positions and velocities of the particles at *any other* time? To answer this question, we shall first try to answer the following simpler question: What predictions can we make about the positions and velocities of the particles at a *slightly different* time t_c, such that the difference $dt = t_c - t_0$ is small enough?

Let us begin by exploring what information we can deduce by starting from the known velocity $\vec{v}_0$ of each particle at the time t_0. (1) Since the velocity $\vec{v}_0$ is defined as $\vec{v}_0 = d\vec{r}/dt$, we can find the displacement $d\vec{r} = \vec{v}_0 dt$ of the particle during the time interval dt. (2) Since $d\vec{r} = \vec{r}_c - \vec{r}_0$, we can then use the known position vector $\vec{r}_0$ of the particle at the time t_0 to find its new position vector $\vec{r}_c = \vec{r}_0 + d\vec{r}$ at the time t_c.

What information can we deduce by starting from the known position of each particle at the time t_0? (1) Since we know the positions and properties of the particles, we can use the known force laws to find at the time t_0 the force on a particle P due to any other particle. (2) By adding the force on P due to *all* the other particles, we can then find at the time t_0 the *total* force $\vec{F}_0$ on P. (3) Since we know the mass m of the particle P we can then use the equation of motion $m\vec{a} = \vec{F}$ to find the acceleration $\vec{a}_0 = \vec{F}_0/m$ of P at the time t_0. (4) Since the acceleration $\vec{a}_0$ is defined as $\vec{a}_0 = d\vec{v}/dt$, we can then find the change $d\vec{v} = \vec{a}_0 dt$ of the velocity of P during the time interval dt. (5) Since $d\vec{v} = \vec{v}_c - \vec{v}_0$, we can then use the known velocity $\vec{v}_0$ of P at the time t_0 to find its new velocity $\vec{v}_c = \vec{v}_0 + d\vec{v}$ at the time t_c.

Our arguments in the preceding two paragraphs lead thus to the following important conclusion:

> If we know the positions and velocities of the particles in an isolated system at any time t_0, we can use the principles of mechanics to find their positions and velocities at a slightly different time $t_0 + dt$. (A-1)

(Note that the time difference dt may be positive or negative, i.e., the time t_c may be later or earlier than the original time t_0.)

To shorten our wording, let us introduce the word "state" defined as follows:

Def. **State of a system**: A specification of the positions and velocities of all particles in the system. (A-2)

Then Rule (A-1) asserts simply that, if we know the state of an isolated system at any time, we can predict the state of the system at a slightly different time. Furthermore, we can use this method of prediction repetitively. For example, suppose that we know the state of the system at some time t_0 and that the time interval $dt = 0.1$ second is small enough. Using this knowledge, we can predict the state of the system 0.1 second later, i.e., at the time $t_0 + 0.1$ second. Then we can, in turn, use this knowledge to predict the state of the system 0.1 second later, i.e., at the time $t_0 + 0.2$ second. Then we can use this knowledge to predict the state of the system 0.1 second later, i.e., at the time $t_0 + 0.3$ second. Proceeding in this way repetitively, we can continue to make successive predictions until we arrive at predictions 10 seconds after t_0, or 10 hours after t_0, or even 100 years after t_0. Thus we see that our previous conclusion, Rule (A-1), implies the following more far-reaching conclusion:

> If we know the positions and velocities of the particles in an isolated system at any time t_0, we can use the principles of mechanics to find their positions and velocities at any other time. (A-3)

In other words, if we know the positions and velocities of any isolated set of particles at any instant, we can predict the entire future history (and

also the entire past history) of the motion of these particles!

DISCUSSION

The preceding conclusion shows that the theory of motion, based on the apparently innocent equation of motion $m\vec{a} = \vec{F}$, has enormous predictive power. Indeed, the conclusion implies that our universe behaves with the perfect predictability and regularity of a giant clockwork. Hence we see why Newton's formulation of the theory of motion was an impressive achievement which has had tremendous influence on scientific and philosophical thought.

The predictive power of the theory can be realized in practice. The many successive calculations required by our method of prediction are merely repetitive additions and multiplications, elementary operations which electronic computers can carry out with great speed. Furthermore, such calculations can often be simplified or circumvented by suitable mathematical methods. As a result, the motion of spaceships, planets, or other astronomical bodies can be predicted with impressive precision.

Newton's theory of motion can be applied successfully to discuss and predict an enormous range of situations of practical interest, both good and evil. For example, it is used to design cars, ships, planes, buildings, bridges, and ballistic missiles; to study the motion of liquids and gases (e.g., the flow of blood in arteries or the motion of the air in the atmosphere); and to design instruments such as the electron microscope. Thus the successful domain of application of the theory ranges from astronomical objects with masses as large as 10^{30} kilogram to atomic particles with masses as small as 10^{-30} kilogram.

Despite its enormous range of applicability, the theory has some limitations because it is based on assumptions which are not valid under some extreme conditions. For example, some basic assumptions of the theory are not valid when particles move so rapidly that their speed is comparable to the speed of light (3×10^8 meter/second). To deal with such conditions, the Newtonian theory of motion must be extended and becomes subsumed under a more general theory, Einstein's theory of relativity. Similarly, other assumptions of the theory may not be valid when particles are as small as atomic size (10^{-10} meter). To deal with such conditions, the Newtonian theory must again be extended and becomes subsumed under a more general theory, quantum mechanics.

Yet, even if we confine ourselves within the vast range of applicability of the Newtonian theory with its equation of motion $m\vec{a} = \vec{F}$, there still remains the important question of how to exploit the inherent predictive power of the theory in practical situations. If we relied merely on fast electronic computers, we could only deal with systems of limited complexity, nor would we acquire the insights necessary for most practical prediction and design. Thus the theory becomes really useful only if we can derive from it general relations and methods allowing us to deal simply with complex situations. Accordingly, we shall begin in this unit to address these questions: What general statements can we make about systems consisting of many particles, without needing to deal with the detailed motion of all the individual particles in such systems? Are there quantities which vary simply or remain unchanged,despite the fact that the positions and velocities of particles change in complicated ways?

SECT.

B EQUATION OF MOTION OF A SYSTEM

Consider a system of interacting particles. This system may also interact with one or more other particles which we shall call "external" particles since they are not part of the system. (For example, the system might consist of the links of a chain. This system may interact with external particles, such as those in the earth and in objects to which the chain is attached.) Can we then use the equation of motion $m\vec{a} = \vec{F}$ for each particle to find an equation describing the motion of the system *as a whole*, without attention to the detailed motion of each particle within the system?

We begin by writing down the equation of motion for each of the particles 1, 2, 3, . . . in the system. Thus

$$m_1\vec{a}_1 = (\vec{F}_{1,2} + \ldots) + \vec{F}_{1,\text{ext}} \qquad \text{(B-1)}$$

$$m_2\vec{a}_2 = (\vec{F}_{2,1} + \ldots) + \vec{F}_{2,\text{ext}} \qquad \text{(B-2)}$$

and so on. The parenthesis on the right side of the equation for each particle is the sum of all *"internal"* forces on this particle, i.e., the sum of all the forces due to all the other particles *in* the system. [For example, in Eq. (B-1) the total internal force on particle 1 could be written in greater detail as $\vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \ldots$.] The last term on the right side of the equation for each particle is the total *"external"* force on the particle, i.e., the sum of all forces on the particle due to all the external particles (denoted by the subscript "ext").

To obtain an equation for the system as a whole, we add the equations of motion Eq. (B-1), Eq. (B-2), ... for all the particles in the system. Thus we obtain

$$m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots = \vec{F}_{\text{int}} + \vec{F}_{\text{ext}} \qquad \text{(B-3)}$$

where we have used the abbreviations

$$\vec{F}_{\text{int}} = \vec{F}_{1,2} + \vec{F}_{2,1} + \ldots \qquad \text{(B-4)}$$

and

$$\vec{F}_{\text{ext}} = \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} + \ldots \qquad \text{(B-5)}$$

The right side of Eq. (B-3) is the "total force on the system," i.e., the sum of all forces on all particles in the system. This total force is the sum of $\vec{F}_{\rm int}$, the "total internal force on the system" (the sum of all internal forces on all particles in the system) and of $\vec{F}_{\rm ext}$, the "total external force on the system" (the sum of all external forces on all particles in the system). But, because of the reciprocal relation between mutual forces, $\vec{F}_{1,2} = -\vec{F}_{2,1}$ so that $\vec{F}_{1,2} + \vec{F}_{2,1} = 0$. Similarly, the sum of any other pair of internal mutual forces is also zero. Hence the total internal force $\vec{F}_{\rm int} = 0$ so that Eq. (B-3) becomes simply:

$$\boxed{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots = \vec{F}_{\rm ext}\,.} \qquad \text{(B-6)}$$

We shall call Eq. (B-6) the "equation of motion of the system." Note that this equation, which describes the motion of the system as a whole, does *not* involve any of the internal forces due to the interactions between particles in the system, but involves only the external forces due to particles outside the system.

Example B-1: Motion of a car and trailer

Consider a car pulling a trailer up a hill. If we focus attention on the system consisting of the car and trailer, the force on the trailer due to the car and the force on the car due to the trailer are *internal* forces. The sum of these mutual forces, i.e., the total internal force on the system, is zero. Thus the total force on the system is simply the total *external* force which is the sum of the gravitational forces on the car and the trailer due to the earth, and of the forces on both these objects due to the road surface.

Finding Total Internal and External Forces (Cap. 2)

B-1 An 80 kg man and his 10 kg parachute form a system of two particles. At one time during a descent, the forces on the parachute are the gravitational force due to the earth, a force of 720 N downward due to the man, and a force of 800 N upward due to the air. At this time, the forces on the man are the gravitational force, a force due to the parachute, and a negligible force due to the air. (a) Which of these forces are internal forces on the system? Which of them are external forces on the system? (b) What is the total internal force on the system? (c) What is the total external force on the system? Use $g = 10\,{\rm m/s}^2$. *(Answer: 105) (Suggestion: [s-10])*

B-2 A 1000 kg car and a 2000 kg truck collide at an icy intersection, where frictional forces due to the road are negligible. At the time of impact the forces on each vehicle are: the gravitational force, the normal force due to the road (equal in magnitude to the gravitational force), and an enormous horizontal force of magnitude 10^5 newton due to the other vehicle. At this time, what are the magnitudes of the total internal force and of the total external force on the system of the colliding vehicles? *(Answer: 104) (Suggestion: [s-3]) (Practice: [p-1])*

SECT.

C CENTER OF MASS

We can express the equation of motion of a system in a compact and familiar form by defining a mass M and an acceleration $\vec{A}$ so as to write Eq. (B-5) in the simple form:

$$\boxed{M\vec{A} = \vec{F}_{\rm ext}\,.} \qquad \text{(C-1)}$$

To assure that this equation is identical with Eq. (B-5), we need only require that

$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots \qquad \text{(C-2)}$$

Provided that this relation is satisfied, we can define M and $\vec{A}$ in any convenient way. Let us then call M the "mass of the system" and define it simply as the sum of the masses of all particles in the system.

Def. | **Mass of a system**: $M = m_1 + m_2 + \ldots$ | (C-3)

Then the quantity $\vec{A}$ is unambiguously defined as the vector such that MA is equal to the right side of Eq. (C-2).

By introducing these definitions of M and $\vec{A}$, the equation of motion Eq. (C-1) of the system looks exactly *as if* it were the equation of motion of a single particle having a mass M and an acceleration $\vec{A}$. This particle would then have a velocity $\vec{V}$ such that $\vec{A} = d\vec{V}/dt$, and correspondingly a position vector $\vec{R}$ such that $\vec{V} = d\vec{R}/dt$. To be consistent with Eq. (C-2), $\vec{V}$ and $\vec{R}$ should then be related to the velocities and position vectors of the individual particles so that *

$$M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots \qquad \text{(C-4)}$$

and

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots \qquad \text{(C-5)}$$

* For example, if we equate the rates of change of both sides of Eq. (C-4), $M(d\vec{V}/dt) = m_1(d\vec{v}_1/dt) + m_2(d\vec{v}_2/dt) + \ldots$, which agrees with Eq. (C-1) if $\vec{A} = d\vec{V}/dt$.

The position of this single equivalent particle is called the "center of mass" (or sometimes the "center of gravity") of the system in accordance

with this definition:

Def. | **Center of mass of a system**: The point having a position vector $\vec{R}$ such that $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots$ | (C-6)

Let us illustrate the preceding comments by considering a few especially simple systems. For example, if the system consists of only one particle, its center of mass is just the position of this particle. The velocity and acceleration of the center of mass are then just those of the particle itself. If the system is uniform and symmetric (e.g., if it is a uniform solid sphere, a basketball, or a uniform circular rod), the Def. (C-6) leads to the result that the center of mass is simply its geometric center. The velocity and acceleration of the center of mass are then just those of the geometric center of the system. Finally, suppose that the system is some composite object consisting of several particles moving together so that each has the *same* velocity $\vec{v}$ and the *same* acceleration $\vec{a}$. (For instance, the composite object might be a sliding sled consisting of several pieces of wood glued together.) Then the velocity $\vec{V}$ of the center of mass is just equal to the velocity $\vec{v}$ of every particle in the composite object, and the acceleration $\vec{A}$ of the center of mass is just equal to the acceleration $\vec{a}$ of every particle in this object.

Since the motion of the center of mass is described by the equation $M\vec{A} = \vec{F}_{\rm ext}$ we can make this statement:

The center of mass of a system moves in the same way as a single particle having a mass M equal to the mass of the whole system and acted on by a force equal to the total external force on the system.	(C-7)

By using this conclusion, we can easily visualize the motion of the center of mass, even if the motion of the system is quite complicated. For example, consider the motion of a twirling baton thrown by a cheerleader. If air friction is neglected, the center of mass of the baton (i.e., the geometrical center of the baton if it is a uniform circular rod) moves then just as simply as a ball subject only to the gravitational force, although the ends of the baton tumble around in a complicated way.

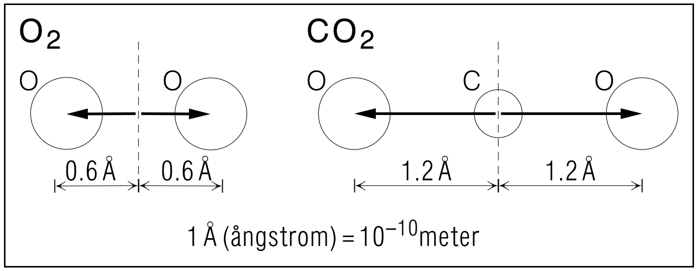


Fig. C-1.

Illustration

C-1 *Finding the center of mass:* To illustrate how the position of the center of mass is found from its definition, Def. (C-6), consider the molecules O_2 and CO_2. For these two systems, each particle is either an oxygen atom, of mass 16 amu, or a carbon atom, of mass 12 amu, where the symbol amu represents the atomic mass unit, defined as (1/12) of the mass of the carbon atom ^{12}C. The position vectors of the atoms in these molecules, measured from an origin at the center of the molecule, are shown in Fig. C-1. (a) What is the position vector $\vec{R}$ of the center of mass of each molecule? (b) Where is the center of mass located relative to the atoms in each molecule? (c) Is there always a particle located at the center of mass? *(Answer: 102) (Suggestion: [s-12])*

Describing Position and Motion of the Center of Mass (Cap. 3)

C-2 Describe the position of the center of mass for these symmetric (or very nearly symmetric) systems: the benzene molecule in Fig. C-2, the air in a spherical soap bubble, the soap film forming the bubble, a solution in a beaker on a chemical balance, a 200 page paperback book, an inflated inner tube. *(Answer: 108)*

C-3 In problem B-1, the 80 kg man and his 10 kg parachute form a system on which the total external force is 100 N downward. The center of mass of this system is located just above the man's head. What is the acceleration $\vec{A}$ of this center of mass? *(Answer: 110)*

C-4 As she passes over the bar in a high-jump, an expert athlete bends sharply backward in a U so that her center of mass (CM) actually passes *under* the bar (Fig. C-3). If the force on her due to the air is negligible, what is the acceleration of the athlete's center of mass as she herself passes over the bar? *(Answer: 101) (Suggestion: [s-2])*

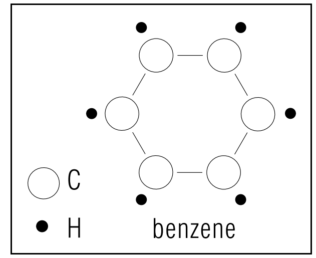


Fig. C-2.

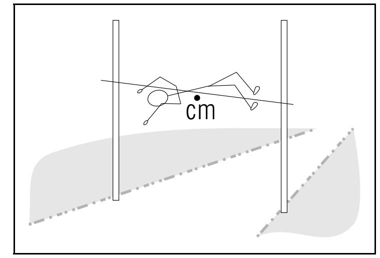


Fig. C-3.

C-5 During the collision described in problem B-2, what is the acceleration of the center of mass of the system consisting of the car and the truck? Is the acceleration of the car or the truck during the collision the same as the acceleration of their center of mass? *(Answer: 107) (Practice: [p-2])*

SECT.

D PROPERTIES OF THE MASS OF A SYSTEM

The equation of motion $M\vec{A} = \vec{F}_{\rm ext}$ of a system allows us to use information about $\vec{A}$ and $\vec{F}_{\rm ext}$ to measure the mass M of the system. This can be done by methods analogous to those used to measure the mass of a single particle. For example, in many cases we can use the fact that the total gravitational force on the system is $m_1\vec{g} + m_2\vec{g} + \ldots = (m_1 + m_2 + \ldots)\vec{g} = M\vec{g}$. It is then possible to measure this gravitational force (and thus the mass M of the system) by comparing it with a known force on the system. For example, we might place the system on a balance so that its center of mass is at rest (i.e., so that $\vec{A} = 0$ and thus $\vec{F}_{\rm ext} = 0$). Then the total gravitational force on the system must be opposite to the measured external force exerted on the system by the balance.

The measured mass of a system provides valuable information since the definition of the mass M of a system, Def. (C-3), implies this conclusion:

> Additive property of mass: The measured mass of a system is equal to the sum of the masses of all the particles in the system. (D-1)

For example, suppose that a system consists of N identical molecules, each having a known mass m. Then the additive property of mass implies that the mass M of the system is $M = Nm$ so that $N = M/m$. Hence it is possible to determine the number N of molecules in the system from a measurement of the mass M of the system.

The preceding comment is especially useful for comparing the numbers of molecules in two different samples of the same substance. Indeed, suppose that we measure the masses M and M' of these two samples. Then we know that the numbers N and N' of molecules in these two samples must be such that $M = Nm$ and $M' = N'm$ (with the *same* mass m of a single molecule). Hence $M/M' = N/N'$. The relative numbers of molecules in the two amounts of the substance can thus be found simply from the relative masses of these two amounts. Since it is easy to measure the masses of systems, but almost impossible to count the enormous numbers of molecules in them, measurements of mass are used in chemistry and daily life whenever information about the number of particles is of central interest. (For example, we are willing to pay twice

as much for 2 grams of gold as for 1 gram because the additive property of mass assures us that we are buying twice as many gold atoms.)

Every ordinary system of particles consists ultimately of “elementary” atomic particles, i.e., of electrons and of the nuclear particles (protons and neutrons) which are the constituents of the atomic nuclei. These atomic particles are ordinarily neither created nor destroyed, and each of them has a definite mass unaffected by its interaction with other particles. Hence the sum of the masses of all the elementary particles in an isolated system, and thus also the measured mass of the system, as given by Rule (D-1), remains unchanged regardless of what processes occur in the system. Thus we arrive at the “principle of conservation of mass” which can be summarized:

> Conservation of mass: The mass of an isolated system remains constant irrespective of all processes. (D-2)

For example, the measured mass of an isolated system remains unchanged irrespective of whether the particles form a solid, or melt to form a liquid, or vaporize to form a gas, or undergo chemical reactions in which molecules dissociate or recombine to form new kinds of molecules.

LIMITATIONS ON VALIDITY

The additive property of the measured mass, Rule (D-1), and the conservation of mass, Rule (D-2), are consequences of the Newtonian theory of motion. Although these principles may seem self-evident, they are *not* valid under conditions where the basic assumptions of the Newtonian theory are not justified. For example, if the particles in a system interact by very strong forces, some of these basic assumptions must be replaced by the refinements of the theory of relativity so that Rule (D-1) and Rule (D-2) are no longer true. The forces between atoms or molecules are sufficiently weak that the Newtonian theory is applicable with great precision in chemistry and everyday life. But the forces between particles within the atomic nucleus are so strong that the principles in Rule (D-1) and Rule (D-2) are only approximately valid in processes involving changing interactions between such particles. [For example, the nucleus of an atom of “heavy hydrogen” (or “deuterium”) consists of a proton of mass m_p and a neutron of mass m_n. But the measured mass M of this nucleus *differs* from the sum $(m_p + m_n)$ of the masses of its constituent particles by more than 0.1 percent, an easily measured amount.]

Knowing About Additivity and Conservation of Mass

D-1 One mole of solid CO_2 (dry ice), which contains 6.0×10^{23} molecules, forms a system of easily-measured mass $M = 0.044\,\mathrm{kg}$. Using this information and the properties of mass, we can find the mass of a carbon dioxide molecule, and the masses of oxygen and carbon atoms, in terms of the unit kilogram. (a) What is the mass m_{CO2} of a carbon dioxide molecule? (b) The mass m_{C} of a carbon atom is known from chemical measurements to be (3/4) of the mass m_{O} of an oxygen atom. Express the mass m_{CO2} of a carbon dioxide molecule in terms of the mass m_{O} of an oxygen atom. (c) Use your results to find the masses m_{O} and m_{C} of the atoms oxygen and carbon. *(Answer: 103)*

SECT.

E MOMENTUM

It is often useful to express the equation of motion of a system of particles directly in terms of their velocities. Let us first show how this can be done in the case of a system consisting of a single particle, and then consider the more complex case of a system consisting of several particles.

SYSTEM CONSISTING OF A SINGLE PARTICLE

The equation of motion of a single particle of mass m is $m\vec{a} = \vec{F}$. By expressing the acceleration $\vec{a}$ in terms of the velocity $\vec{v}$,we can write

$$m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}, \qquad \text{where } \vec{p} = m\vec{v} \qquad \text{(E-1)}$$

Here we have used the fact that $d(m\vec{v}) = md\vec{v}$ (since m is merely a constant) and have introduced the convenient abbreviation $\vec{p} = m\vec{v}$. This vector $\vec{p}$ is called the "momentum" of the particle. Then we can write the equation of motion of a single particle either as

$$m\vec{a} = \vec{F}$$

or

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad \text{(E-2)}$$

According to its definition, the momentum $\vec{p}$ of a single particle is simply the vector obtained by multiplying the mass m of the particle by its velocity $\vec{v}$. Since $d\vec{p}/dt = \vec{F}$, the rate of change of the momentum of a particle is equal to the total force on this particle. Thus the total force on a particle is large if the rate of change of its momentum $\vec{p} = m\vec{v}$ is large, i.e., if the particle has a large mass m and if its velocity $\vec{v}$ changes rapidly.

For example, if a watch is dropped on the floor, the injury sustained by the watch depends on the total force $\vec{F}$ acting on the watch during its impact with the floor. This force $\vec{F}$, which is equal to the rate of change of the momentum $\vec{p} = m\vec{v}$ of the watch, will be large if the watch has a large mass m and if its velocity $\vec{v}$ changes rapidly (i.e., if the ground surface is hard so that the watch is brought to rest during a very short time after hitting the surface).

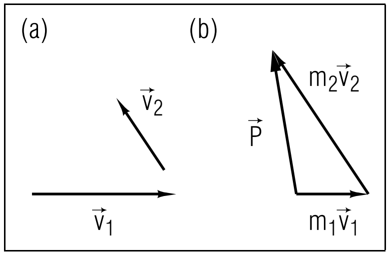


Fig. E-1: Momentum of a system of two particles. (a) Velocities of the particles. (b) Momenta of the particles and momentum $\vec{P}$ of the system. (The particle masses are such that $m_2 = 4m_1$.)

GENERAL SYSTEM

Consider now a system consisting of several particles. According to Eq. (B-6), its equation of motion is

$$m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots = \vec{F}_{\rm ext} \qquad \text{(E-3)}$$

By using the result Eq. (E-1) for each particle, this relation can be written as

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \ldots = \frac{d(\vec{p}_1 + \vec{p}_2 + \ldots)}{dt} = \vec{F}_{\rm ext} \qquad \text{(E-4)}$$

where we have used the fact that the rate of a sum is equal to the sum of the rates. Then we can write Eq. (E-4) in the simple form

$$\boxed{\frac{d\vec{P}}{dt} = \vec{F}_{\rm ext}} \qquad \text{(E-5)}$$

if we introduce the convenient abbreviation $\vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots$. This vector $\vec{P}$, called the "momentum of the system," is simply the sum of the momenta of all the particles in the system. (See Fig. E-1.)

Def. **Momentum of a system**: $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots$ (E-6)

By using Eq. (C-4), this definition of the momentum of a system can also be written as

$$\vec{P} = M\vec{V} \qquad \text{(E-7)}$$

In other words, the momentum of a system is the vector obtained by multiplying the mass M of the system by the velocity $\vec{V}$ of its center of mass. (Thus the momentum of a system is the same as that of a single

particle having a mass M equal to the mass of the system and a velocity $\vec{V}$ equal to that of the center of mass of the system.)

The relation $d\vec{P}/dt = \vec{F}_{\rm ext}$ of Eq. (E-5) summarizes the equation of motion of a system in a simple and useful form. It asserts that the rate of change of momentum of any system is equal to the total external force on it. Thus this external force is large if the mass M of the system is large and if the velocity $\vec{V}$ of its center of mass changes rapidly.

FORCES PRODUCED IN COLLISIONS

Consider a person who collides with a surface (e.g., a person hitting the ground after falling from some great height). Then the velocity $\vec{V}$ of the person's center of mass changes very rapidly as the person is brought to rest during a very short time after his impact with the surface. Hence the rate of change of the person's momentum is quite large, especially if he has a large mass M. Consequently the total external force $\vec{F}_{\rm ext} = d\vec{P}/dt$ exerted on the person during the collision is also large, indeed often large enough to cause serious injuries. *

> * The total external force on the person during the collision consists mostly of the contact force exerted on the person by the surface, since this force is usually much larger than any other forces (such as gravitational forces).

To estimate the magnitude of the external force acting on the person during the collision, suppose that the time interval dt during which the collision occurs is small enough so that the rate of change of momentum $d\vec{P}/dt$, and thus the force $\vec{F}_{\rm ext}$, is nearly constant during this time interval. Then the equation of motion of the person implies that $\vec{F}_{\rm ext} = d\vec{P}/dt$ where $\vec{P} = M\vec{V}$ is the momentum of the person of mass M when the velocity of his center of mass is $\vec{V}$. If the initial value of this velocity just before the collision is $\vec{V}_0$ and the person is brought to rest as a result of the collision (so that $\vec{V} = 0$ just after the collision), the change in the person's momentum during the collision is then

$$d\vec{P} = 0 - M\vec{V}_0 = -M\vec{V}_0$$

Hence the equation of motion of the person leads to the result

$$\vec{F}_{\rm ext} = \frac{d\vec{P}}{dt} = -\frac{M\vec{V}_0}{dt} \qquad \text{(E-7)}$$

What can be done to keep the total external force on the person during the collision as small as possible? From Eq. (E-7) it is apparent that it helps if the mass M of the person is small and if his center-of-mass speed $\vec{V}_0$ just before the impact is small. Furthermore, one can try to make the duration dt of the impact as long as possible. For example, a falling person landing on the ground with his feet can lengthen the time interval dt during which his center of mass is brought to rest by allowing his knees to bend and thus breaking his fall. Similarly, persons have survived falls from very great heights in cases where they fell on soft snow (so that the time interval during which they were brought to rest was several times longer than in a collision with the hard ground).

Understanding the Definition of Momentum (Cap. 1a)

E-1 *Statement and example:* Consider once again our example of a car (of mass $m_c = 1000\,\text{kg}$) and a truck (of mass $m_t = 2000\,\text{kg}$) which collide at an icy intersection. Just before the collision, the car has a velocity $\vec{v}_c = 20\,\text{m/s}$ north and the truck has a velocity $\vec{v}_t = 10\,\text{m/s}$ east. Define by an equation and then find a value for the momentum $\vec{P}$ of each of these systems: (a) the system consisting of the car alone, (b) the system consisting of the truck alone, (c) the system consisting of both the car and the truck. *(Answer: 109) ([s-13], [p-3])*

E-2 *Comparing momentum and velocity:* Which of the following statements correctly compare momentum and velocity? If a statement is incorrect, briefly explain why. (a) If a single particle's speed or direction of motion changes, both its velocity and its momentum also change. (b) Two particles having the same velocity must have the same momentum. (c) The direction of the momentum of any system is the same as the direction of the velocity of its center of mass. (d) The momentum of a system of several particles can be zero even though the velocities of these particles are not zero. *(Answer: 106) (Suggestion: [s-11])*

E-3 *Relating momentum to velocity for composite systems:* The car and truck described in problem E-1 lock together during the collision. Immediately afterward, the momentum of the composite wreckage is $2.8 \times 10^4\,\text{kg m/s}$ northeast. What is the velocity of the wreckage (i.e., of its center of mass)? *(Answer: 123)*

E-4 *Relating momentum to the velocity of part of a system:* When a patient lies on a light cot which is free to move horizontally, his body moves (with the cot) minutely back and forth. This body motion,

measured by the *ballistocardiogram*, is due to the motion of the blood pumped by the heart. As a simplified example illustrating the relation of the ballistocardiogram to blood flow, imagine the patient's body as a system with two parts: (1) the volume of blood, of mass $m_1 = 0.08\,\text{kg}$, ejected from the heart in every stroke, and (2) the rest of the patient's body, of mass $m_2 = 80\,\text{kg}$. This system has momentum $\vec{P} = 0$. When the volume of blood is ejected into the aorta, its center of mass moves with some velocity $\vec{v}_1$. Correspondingly, the center of mass of the rest of the patient's body moves with a typical measured velocity $\vec{v}_2 = (5 \times 10^{-4}\,\text{m/s})\hat{x}$, where the direction $\hat{x}$ points from the patient's head toward his feet. (a) What is the velocity $\vec{v}_1$ of the center of mass of the ejected blood? (b) *Review:* The center of mass of the ejected blood and that of the rest of the body continue to move with these velocities for about 0.1 second. How far, and in what direction, does each center of mass move during this time? *(Answer: 116) (Suggestion: [s-9])*

Understanding the Relation $d\vec{P}/dt = \vec{F}_{\text{ext}}$ (Cap. 1b)

E-5 *Example:* At the top of its trajectory, a baseball of mass 0.16 kg is moving horizontally with a speed of 15 m/s. (a) If the force on the baseball due to the air is negligible, what is the total external force $\vec{F}_{\text{ext}}$ on the baseball? (b) What is the change $d\vec{P}$ in its momentum during the small time interval $dt = 2.0$ second? (As long as $\vec{F}_{\text{ext}}$ is constant, any time interval is small enough.) *(Answer: 112)*

E-6 *Comparing momentum and momentum change:* (a) For the example in problem E-5, draw a rough diagram showing the baseball's momentum $\vec{P}$ at the top of its trajectory, the subsequent change $d\vec{P}$ in its momentum, and its new momentum $\vec{P}\prime = \vec{P} + d\vec{P}$. Are the values of any of these quantities the same? (b) The total external force on a bowling ball rolling down an alley is negligible. As the ball rolls, is its momentum zero? Is the change in its momentum zero? *(Answer: 118)*

E-7 *Relating quantities:* Because of air friction, an 80 kg man who falls from a great height will have a speed of only 50 m/s (the "terminal velocity") when he hits the ground. (a) What is the change in the momentum of such a man as he comes to rest during the impact with the ground? (b) Suppose the total external force on the man during the impact has an approximately constant value of $1.2 \times 10^5\,\text{N}$ upward, the maximum force he can sustain without serious injury if he lands on his back. What is the corresponding minimum duration of impact needed to avoid serious injury? (A parachutist whose parachute failed to open once experienced an impact of this duration on soft snow. He suffered only two minor fractures and a few bruises.) *(Answer: 114) (Suggestion: [p-4])*

E-8 *Dependence of total external force on momentum change:* Consider two persons of different mass who fall from the same height and thus hit the ground with the *same* velocity. Suppose that both take the *same* small enough time interval to come to rest. (a) During the impact, is the momentum change of the more massive person larger than, equal to, or smaller than the momentum change of the less massive one? (b) Is the total external force during impact (and thus the chance of injury) the same for the two persons? If not, which one experiences the larger $\vec{F}_{\rm ext}$? *(Answer: 120) (Suggestion: [s-6])*

SECT.

F CONSERVATION OF MOMENTUM

Suppose that the total external force on a system is zero. Then its equation of motion $d\vec{P}/dt = \vec{F}_{\rm ext} = 0$. Since the rate of change of the momentum $\vec{P}$ of the system is zero, $\vec{P}$ must then remain constant. Thus we arrive at this conclusion, known as the "principle of conservation of momentum":

Conservation of momentum: If the total external force on a system is zero, the momentum of the system remains constant. * (F-1)

* Similarly, if the component vector of $\vec{F}_{\rm ext}$ parallel to some direction is zero, the component vector of $\vec{P}$ parallel to this direction must remain constant.

For example, consider a system consisting of two interacting particles having masses m_1 and m_2 (e.g., a car and a truck colliding at an icy intersection). Suppose that the total external force on this system is zero. Then the conservation of momentum states that

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant} \tag{F-2}$$

Thus the interacting particles move always in such a way that the sum of their momenta remains unchanged, i.e., so that any change in the momentum $m_1\vec{v}_1$ of one of the particles is always accompanied by an opposite change of the momentum $m_2\vec{v}_2$ of the other particle. Correspondingly, the velocity change $\Delta\vec{v}_1$ of one of the particles has during any time a direction opposite to the velocity change $\Delta\vec{v}_2$ of the other particle (although the magnitudes of these velocity changes are different unless the masses m_1 and m_2 of the particles are the same). *

* This relation between velocity changes is consistent with the relation $m_1\vec{a}_1 = -m_2\vec{a}_2$ connecting the accelerations of two particles isolated from the rest of the universe.

If we consider the preceding two particles at any two times t and t', the conservation of momentum implies that

$$\vec{P} = \vec{P}\prime \tag{F-3}$$

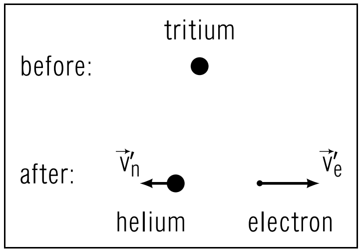


Fig. F-1: Radioactive disintegration of a tritium nucleus.

or equivalently that

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}\prime_1 + m_2\vec{v}\prime_2 \qquad \text{(F-4)}$$

where we have denoted quantities at the time t by unprimed symbols and quantities at the time t' by primed symbols.

The conservation of momentum is especially useful for comparing the motions of particles before and after some process. For example, such a process might be a "collision," i.e., a process in which particles interact (by contact or long-range forces) for some time, without interacting either before or afterwards. Alternatively, such a process might be the dissociation or recombination of particles, as illustrated in this example:

Example F-1: Recoil of a radioactive nucleus

Tritium is a radioactive hydrogen atom which can be substituted for some ordinary hydrogen atoms in molecules of biological interest (such as DNA). In this way one can label particular atoms in such molecules and can thus study biological processes at the molecular level. The tritium atom is radioactive because its nucleus disintegrates into a helium nucleus by emitting a high-speed electron. Suppose that the external forces on a tritium nucleus are negligible. If this nucleus is initially at rest and then emits an electron of mass $m_e = 9 \times 10^{-31}$ kg with a speed of 8×10^7 m/s, what is the "recoil velocity" acquired by the remaining helium nucleus of mass $m_n = 5 \times 10^{-27}$ kg? (This recoil velocity is partially responsible for the damage done to the biological molecule in which the tritium atom is incorporated.)

Description: Fig. F-1 illustrates the situation before and after the disintegration of the tritium nucleus. Using primes to indicate quantities *after* the disintegration, let us call $\vec{v}\prime_e$ the velocity of the electron and $\vec{v}\prime_n$ the velocity of the helium nucleus. Then we know the masses m_e and m_n, and the speed $|\vec{v}\prime_e| = 8 \times 10^7$ m/s. We should like to find the velocity $\vec{v}\prime_n$.

Planning: Since the total external force on the system is negligible, the conservation of momentum implies that $\vec{P} = \vec{P}\prime$ where $\vec{P}$ is the momentum of the system before the disintegration and $\vec{P}\prime$ is its momentum after the disintegration. Here $\vec{P} = 0$ since the original tritium nucleus is at rest. Then the relation $\vec{P} = \vec{P}\prime$ is equivalent to

$$0 = m_e\vec{v}\prime_e + m_n\vec{v}\prime_n$$

Implementation: We can solve the preceding equation for $\vec{v}\prime_n$. Thus $m_n\vec{v}\prime_n = -m_e\vec{v}\prime_e$ or

$$\vec{v}\prime_n = -\frac{m_e}{m_n}\vec{v}\prime_e$$

Hence the velocity $\vec{v}\prime_n$ of the helium nucleus has a direction *opposite* to that of the velocity $\vec{v}\prime_e$ of the emitted electron. (This is why the nucleus is said to "recoil.") Furthermore the magnitude of this velocity is

$$|\vec{v}\prime_n| = \frac{m_e}{m_n}|\vec{v}\prime_e| = \frac{9 \times 10^{-31}\,\text{kg}}{5 \times 10^{-27}\,\text{kg}}(8 \times 10^7\,\text{m/s}) = 1.4 \times 10^4\,\text{m/s}$$

Checking: Note that the speed of the helium nucleus is much less than that of the electron. This makes sense since this nucleus has a much larger mass, and thus always a much smaller acceleration, than the electron.

Understanding Conservation of Momentum (Cap. 1c)

F-1 *Applicability:* To which of the following systems does the principle of conservation of momentum apply? (a) A heart patient and the cot he lies on, when this system is supported by air jets so that it can move horizontally with negligible friction. (b) Two vehicles involved in a collision on a level icy intersection where frictional forces due to the road are negligible. (c) One of the vehicles in this collision. (d) A rocket and its exhaust gases when both are far from other objects. (e) The preceding system near the earth. *(Answer: 111) (Suggestion: [s-8])*

Now: Go to tutorial section F.

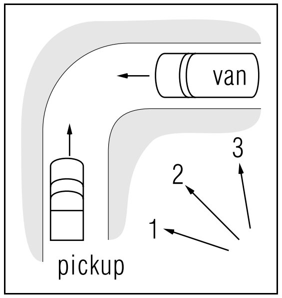


Fig. F-2.

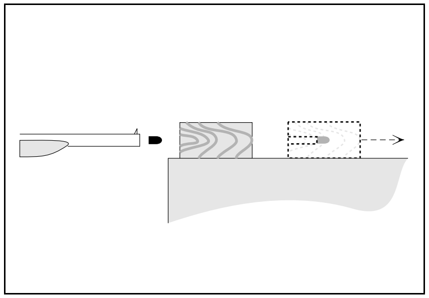


Fig. F-3.

Applying Conservation of Momentum (Cap. 1a, 4)

F-2 A 1500 kg pickup truck and a 3000 kg moving van collide on a level icy curve (Fig. F-2). Just before the collision, both trucks are traveling with equal speeds of 20 m/s in perpendicular directions. Frictional forces on the tires of the vehicles are negligible during the collision. (a) The two vehicles lock together in the collision. Which of the directions in Fig. F-2 best indicates the direction of the momentum (or velocity) of the combined wreckage just after the collision? (b) What is the magnitude of this momentum? (c) What is the speed of the wreckage just after the collision? *(Answer: 113) (Suggestion: [s-1])*

F-3 Figure F-3 illustrates a simple method for measuring the large "muzzle velocity" of a bullet emerging from a rifle. The bullet, of mass 1 gram, immediately strikes a stationary wood block of mass 1 kg, and the two then slide together with a small, easily-measured speed along a nearly frictionless horizontal surface. During and after this collision, the total external force on the system of bullet and block is thus negligible. (a) Suppose the measured speed of the block and embedded bullet is 0.5 m/s. What was the momentum of the system of bullet and block just before their collision? (b) What was the speed of the bullet at this time? *(Answer: 119)*

(Practice: [p-5])

SECT.
G SUMMARY

DEFINITIONS

state of a system; Def. (A-2)

total external force; Eq. (B-5)

mass of a system; Def. (C-3)

center of mass; Def. (C-6)

momentum; Def. (E-6)

IMPORTANT RESULTS

Predictive power of mechanics: Rule (A-3)

Knowledge of the state of a system at any time allows prediction of its state at any other time.

Equation of motion of a system: Eq. (C-1), Eq. (E-5)

$M\vec{A} = \vec{F}_{\rm ext}$ or $d\vec{P}/dt = \vec{F}_{\rm ext}$, where $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots = M\vec{V}$ is momentum of system, $\vec{A}$ and $\vec{V}$ are acceleration and velocity of its center of mass.

Properties of the mass of a system: Rule (D-1), Rule (D-2)

$M = m_1 + m_2 + \ldots$

M for isolated system remains constant

Conservation of momentum: Rule (F-1)

If $\vec{F}_{\rm ext} = 0$, $\vec{P} =$ constant.

NEW CAPABILITIES

You should have acquired the ability to:

(1) Understand these relations for a system of particles:
(a) the definition of momentum (Sec. E; [p-3]),
(b) the relation $d\vec{P}/dt = \vec{F}_{\rm ext}$ (Sec. E; [p-4]),
(c) the conservation of momentum (Sec. F).

(2) Use the forces acting on the particles in a system to find the total internal and external force on the system. (Sec. B; [p-1])

(3) (a) Use the external force on a system of particles to find the acceleration of its center of mass. (b) Locate the position of the center of mass for symmetric systems. (Sec. C; [p-2])

(4) By applying conservation of momentum, use the masses and velocities of the particles in a system at one time to find the momentum of the system at any time. (Sec. F; [p-5])

Describing the Motion of a System (Cap. 1, 3, 4)

G-1 The common squid *Loligo vulgaris* maneuvers by jet propulsion. The adult has a mass of 150 gram with its mantle full of water, and it forcefully expels about 50 gram of water from its mantle in one jet pulse. In a filmed sequence of an escape response, such an adult squid accelerated from rest to a speed of 2 m/s with one pulse. To estimate the speed of the water expelled in this pulse, assume that the total external force on the system of squid and expelled water is negligible. (a) What is the momentum of this system immediately after the pulse? (b) What is the speed of the expelled water (i.e., of its center of mass) at this time? *(Answer: 117)*

G-2 Because the brain is easily damaged by large accelerations of the head, blows to the head are a serious hazard in heavyweight boxing. To illustrate this point, consider the typical motion of a boxer's head, of mass 4.0 kg, struck squarely by an opponent. Just before the blow, the center of mass of the boxer's head has a velocity of 1.0 m/s along a horizontal direction $\hat{x}$ toward the opponent, and just afterwards it has a recoil velocity of 8.0 m/s in the opposite direction. The duration of the blow is about 0.015 second. (a) What is the change in the momentum of the boxer's head during the blow? (b) Assuming that the total external force on the boxer's head is approximately constant during the blow, what is the value of this force? (c) What is the acceleration of the center of mass of the boxer's head during the blow? *(Answer: 115)*

G-3 An unknown radioactive atom, isolated and initially at rest, decays by emitting a particle. The resulting "daughter" atom is identified as a lead isotope (^{210}Pb) of mass $210\,m_0$, where m_0 is the mass of a proton. After the decay, the lead atom has a speed of 3.4×10^5 m/s, and the emitted particle has a speed of 1.8×10^7 m/s. (a) At this time, what is the momentum of the system consisting of lead atom and emitted particle? (b) The original radioactive atom can be identified if the mass of the emitted particle, and thus its identity, is known. What is the mass of this particle, expressed in terms of the mass m_0 of a proton? *(Answer: 128)*

SECT.

H PROBLEMS

H-1 *Severity of elastic and inelastic impacts :* Depending on the design of an automobile dashboard, a head hitting the dashboard in a collision might either bounce or come to rest during the impact. Let us compare the severity of these two types of impact if the person's head has a mass of 4.0 kg and hits the dashboard with a typical speed of 20 m/s. (a) Suppose the head bounces with a velocity just after impact that is equal in magnitude but opposite in direction to its velocity just before impact. (Such an impact is called "elastic.") If the impact lasts 0.010 second, what is the magnitude $F_{\rm ext}$ of the total external force on the head, assuming that this force is nearly constant during the impact? (b) Suppose instead that the head comes to rest in an impact of the same duration. (Such an impact is called "inelastic.") What is the magnitude $F_{\rm ext}$ of the total external force exerted on the head in this case? (c) Which of these types of impact is likely to cause the most severe head injury? *(Answer: 125) ([s-7], [p-6])*

H-2 *Center of mass displacement and impact force:* As an object comes to rest during an impact, its center of mass moves through some displacement of magnitude ℓ. This distance, which is often easier to measure than the duration Δt of the impact, can be related directly to the magnitude $F_{\rm ext}$ of the total external force on the object if we assume that $\vec{F}_{\rm ext}$ is constant during the impact. Thus the object's center of mass moves with constant acceleration (i.e., uniformly decreasing speed) during the impact. (a) First, write an expression for $F_{\rm ext}$ in terms of the object's mass M, the speed V_0 of its center of mass just before impact, and the duration Δt of the impact. (b) Now express the distance ℓ moved by the object's center of mass during the impact in terms of V_0 and Δt. (c) By combining your results, express $F_{\rm ext}$ in terms of M, V_0 , and ℓ alone. (d) To illustrate, consider a 100 kg man who jumps to the ground from the small height of 2 meter, thus hitting the ground feet first with a speed of 6 m/s. His ankles are likely to break if $F_{\rm ext}$ exceeds 10^5 N. Find the value of $F_{\rm ext}$ if he lands with his legs stiff, so that $\ell = 1$ cm, and if he bends his knees on landing, so that $\ell = 10$ cm. Can he break his ankles in such a small jump? *(Answer: 122) ([s-5], [p-7])*

H-3 *How far you can safely drop if you land stiff-legged:* As a person of mass M drops feet first from a small height h to the ground, the person's center of mass falls from rest a distance h with acceleration

$\vec{g}$, and thus acquires some speed V_0 just before the person hits the ground. (a) Express V_0 in terms of h and g. (b) Using the results of problem H-2, show that the magnitude of the total external force on the person during impact is given by $F_{\rm ext} = Mgh/\ell$, where ℓ is the distance moved by the person's center of mass during impact. (c) Assuming that 10^5 N is the maximum safe value for $F_{\rm ext}$ what is the maximum distance h an 80 kg man and a 50 kg woman can safely drop if they land stiff-legged, so that $\ell = 1$ cm for both? *(Answer: 127) (Suggestion: [s-4])*

H-4 *Estimating the impact force in a rear-end collision:* A 1000 kg car moving with a velocity of 10 m/s north hits the back of a 3000 kg truck at rest at a stoplight. Just after the collision, which lasts 0.02 second, the truck has a velocity of 4 m/s north. (a) Assuming that frictional forces due to the road are negligible during the collision, what is the velocity of the car just after the collision? (b) What is the total external force exerted on the *car* during the collision, assuming that this force is nearly constant? What is the total external force on the *truck* during the collision? *(Answer: 124)*

H-5 *Deflection of atoms in ion production:* To produce negatively-charged oxygen ions (O^-) used in an experiment, oxygen atoms of mass 2.7×10^{-26} kg moving in vacuum with a speed of 1.5×10^3 m/s are struck by electrons of mass 1.0×10^{-30} kg moving in a perpendicular direction with a speed of 3.0×10^7 m/s. When an electron strikes and combines with an oxygen atom, the resulting ion is deflected from the path of the oxygen atoms. For such a collision, find (a) the angle between the oxygen ion's velocity and the original direction of motion of the oxygen atom, and (b) the speed of the ion. External forces on this system of colliding particles are negligible. *(Answer: 121)*

H-6 *Nabbing speeders with momentum conservation:* Suppose your investigations of an accident reveal that a 3000 kg truck was traveling north with a speed of 10 m/s when it was hit by a 1000 kg car traveling west, and that the combined wreckage moved precisely northwest from the point of impact. To decide whether to add speeding to the list of the car driver's offenses, find the speed of the car just before the collision, assuming that frictional forces on the vehicles were negligible during the collision. *(Answer: 126)*

TUTORIAL FOR F

RELATING MASSES AND VELOCITIES USING MOMENTUM CONSERVATION

f-1 *PURPOSE:* Whenever the total external force on a system of particles is negligible or zero, we can use the principle of conservation of momentum to relate the masses and velocities of particles in the system at two different times. To do so, we use the known masses and velocities of particles in the system at one time t to find its momentum $\vec{P}$ at this time. Using this value, and the fact that the system's momentum $\vec{P}\prime$ at any other time t' must be equal to $\vec{P}$, we can find the desired information about the masses or velocities of the particles in the system at the time t'.

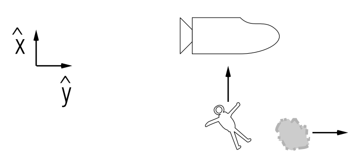
The purpose of the next frame is to illustrate this method in more detail by applying it to a problem. In doing so, we shall follow the basic steps of the problem-solving strategy outlined in text section D of Unit 409.

f-2 *A METHOD FOR APPLYING CONSERVATION OF MOMENTUM:* Let us systematically solve this problem:

A 100 kg astronaut, isolated in deep space, uses a container of gas to maneuver near his spacecraft. Shortly after the astronaut releases a 0.10 kg burst of gas from the container, he has a velocity of 1.0 m/s along a direction $\hat{x}$ toward his spacecraft, while the center of mass of the gas burst has a velocity of 1700 m/s along a perpendicular direction $\hat{y}$. What was the astronaut's velocity *before* he released the burst?

DESCRIPTION

Sketch:



Known information: After the burst (time t): astronaut, of mass $m_a = 100\,\text{kg}$, has velocity $\vec{v}_a = 1.0\,\text{m/s}\,\hat{x}$; gas burst, of mass $m_g = 0.1\,\text{kg}$, has velocity $\vec{v}_g = 1700\,\text{m/s}\,\hat{y}$. Before the burst (time t'): astronaut and gas, of mass $M = m_g + m_a = 100\,\text{kg}$, move together. Astronaut and gas interact, but are otherwise isolated. *Desired information:* velocity $\vec{v}\prime_a$ of astronaut before releasing the burst.

PLANNING

(1) *Decide whether conservation of momentum applies to a system.*

To do so, we must find a system on which the total external force is zero or negligible. This is not true of a system consisting of either the astronaut or gas burst individually, since these objects interact with each other. However, the system consisting of *both* the astronaut and gas burst is isolated, so that conservation of momentum applies to this system.

(2) *Express the principle of conservation of momentum in terms of symbols for known and desired information.*

As a first step, we write $\vec{P} = \vec{P}\prime$, where $\vec{P}$ is the momentum of the system of astronaut and gas *after* the release of the burst (which we can find from known information), and $\vec{P}\prime$ is the unknown momentum of this system *before* the release of the burst. Using the symbols we have introduced, we can write the relations $\vec{P} = m_a\vec{v}_a + m_g\vec{v}_g$ and $\vec{P}\prime = M\vec{v}\prime_a$. Since we know all the quantities in these relations except the desired velocity $\vec{v}\prime_a$, we can find this velocity.

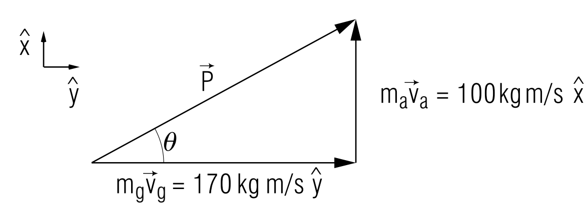
IMPLEMENTATION

(1) *Solve algebraically for the desired quantity.*

Since $\vec{P}\prime = M\vec{v}\prime_a = \vec{P}$, $\vec{v}\prime_a = \vec{P}/M$. We can find the momentum $\vec{P}$ from the relation $\vec{P} = m_a\vec{v}_a + m_g\vec{v}_g$.

(2) *Substitute known values, and find the desired quantity.*

We first use a vector diagram to construct the vector $\vec{P}$:



Using this diagram, we can find the magnitude and direction of $\vec{P}$. Thus $P = \sqrt{(1.0 \times 10^2\,\mathrm{kg\,m/s})^2 + (1.7 \times 10^2\,\mathrm{kg\,m/s})^2}$, or $P = \sqrt{3.9 \times 10^4\,(\mathrm{kg\ m/s})^2} = 2.0 \times 10^2\,\mathrm{kg\,m/s}$. We can find the angle θ using the relation $\sin\theta = (100\,\mathrm{kg\,m/s})/P = 0.50$, so that $\theta = 30°$. Thus $\vec{P} = 2.0 \times 10^2\,\mathrm{kg\,m/s}$ at an angle of 30° from $\hat{y}$. Thus, since $M = 100\,\mathrm{kg}$, $\vec{v}\prime_a = \vec{P}/M = 2.0\,\mathrm{m/s}$ at an angle of 30° from $\hat{y}$.

CHECKING

Our work is correct, and the result has the correct unit and a reasonable magnitude and direction.

The method we have illustrated is useful in applying conservation of momentum to any problem. In particular, it should help you solve systematically the remaining problems in text section F.

Now: Go to text problem F-2.

PRACTICE PROBLEMS

p-1 *FINDING TOTAL INTERNAL AND EXTERNAL FORCES (CAP. 2):* While working on the outside of a 1000 kg spacecraft orbiting near the earth's surface, a 100 kg astronaut holds on to the spacecraft's hull, thus exerting a force of magnitude 1 N on the spacecraft. The gravitational forces on the astronaut and spacecraft, which cause them to orbit together, have about the same magnitude as on the earth's surface. What are the magnitudes of the total external force and the total internal force exerted on the system of astronaut and spacecraft? *(Answer: 5) (Suggestion: review text problems B-1 and B-2.)*

p-2 *DESCRIBING POSITION AND MOTION OF THE CENTER OF MASS (CAP. 3):* A weightlifter's barbell consists of a 10 kg rod which holds four wheel-shaped weights, one of mass 50 kg and one of mass 20 kg at each end. (a) Where is the barbell's center of mass located? (b) During a lift, the weightlifter exerts a force of 1515 N upward on the rod. At this time, what is the total external force acting on the barbell? What is the acceleration of the barbell's center of mass? *(Answer: 2) (Suggestion: review text problems C-2 through C-5.)*

p-3 *UNDERSTANDING THE DEFINITION OF MOMENTUM (CAP. 1A):* During a double-play, a baseball player of mass 80 kg jumps vertically upward to catch a ball thrown over his head. Just before the catch, the 0.15 kg ball has a horizontal velocity of 20 m/s to the right, while the player has a velocity of 0.05 m/s upward. At this time, what is the magnitude of the momentum of the system of player and ball? *(Answer: 9) Now: Return to text problem E-1 and be sure your work is correct.*

p-4 *UNDERSTANDING THE RELATION $D\vec{P}/DT = \vec{F}_{EXT}$ (CAP. 1B):* Just before it is hit by a bat, a baseball of mass 0.15 kg has a horizontal velocity of 30 m/s toward the bat. Just after it is hit, the baseball has a horizontal velocity of 34 m/s in the opposite direction, or away from the bat. (a) What is the change in the baseball's momentum during its impact with the bat? (b) This impact lasts only 0.001 second. Assuming that this interval is small enough, what is the total external force on the baseball during impact? (c) To show that this force is nearly all due to the bat, find the ratio of its magnitude to the magnitude of the gravitational force on the baseball. *(Answer: 1) (Suggestion: review text* *problem E-7.)*

p-5 *APPLYING CONSERVATION OF MOMENTUM (CAPS. 1A, 4):* An alpha particle (helium nucleus) of mass 6.7×10^{-27} kg, moving with a speed of 2.0×10^{7} m/s along a direction $\hat{x}$, collides head-on with an unknown nucleus at rest in the gas filling a "cloud chamber." Immediately after the collision, the alpha particle moves back along its original path with a speed of 1.0×10^{7} m/s,while the nucleus moves along the original direction of motion $\hat{x}$ of the alpha particle with a speed of 9.0×10^{6} m/s. These data, which are obtained from observations of condensed droplets indicating the tracks of the particles in the cloud chamber, can be used to find the mass of the unknown nucleus. (a) What is the momentum of the system of alpha particle and nucleus after their collision? (This system is isolated.) (b) What is the mass of the unknown nucleus? *(Answer: 6) (Suggestion: review the procedure followed in tutorial frame [f-2].)*

More Difficult Practice Problems (Text Section H)

p-6 *SEVERITY OF AUTO COLLISIONS:* Consider the following possible collisions of a car of mass m traveling north with a speed v: (1) the car strikes a cliff head-on and comes to rest, (2) the car collides with an identical car at rest, and both cars move north together with a speed $v/2$, (3) the car collides head-on with an identical car moving south with speed v, so that both cars come to rest in the collision. (a) Assuming that these collisions all have the same small enough duration dt, write an expression for the magnitude $F_{\rm ext}$ of the total external force on the car in each collision. (b) Rank these collisions in order of increasing severity (i.e., in order of increasing $F_{\rm ext}$). *(Answer: 8) (Suggestion: review text problem H-1.)*

p-7 *IMPACT OF A GOLF CLUB WITH A GOLF BALL:* In the early days of high-speed photography, pictures were taken of a golf club hitting a stationary golf ball of mass 0.06 kg. After an impact of duration 2×10^{-4} second, the ball's center of mass was found to have a speed of 60 m/s. (a) Assuming that the total external force $\vec{F}_{\rm ext}$ on the ball was nearly constant during the impact, estimate its magnitude. (b) Under the same assumption, estimate the distance traveled by the ball's center of mass during the impact. *(Answer: 4) (Suggestion: review text problem H-2.)*

SUGGESTIONS

s-1 *(Text problem F-2):* Use the procedure outlined in tutorial frame [f-2] as a guide.

s-2 *(Text problem C-4):* Only one external force, the gravitational force due to the earth, acts on the particles in the high-jumper's body. Thus the equation of motion of her center of mass is $M\vec{A} = \vec{F}_{\rm ext} = M\vec{g}$, where M is her mass.

s-3 *(Text problem B-2):* Note that there are only two external forces on each of the particles in this system (the truck and the car), and that these forces are equal in magnitude and opposite in direction.

s-4 *(Text problem H-3):* Assume that the person takes a time interval Δt to hit the ground. Since the person's center of mass moves with constant acceleration $\vec{g}$, its motion is described by these relations discussed in Unit 406: $\Delta\vec{v} = \vec{g}\Delta t$ and $\Delta r = \vec{v}_A\Delta t + 1/2\vec{g}(\Delta t)^2$. Use these relations to express the final speed v_0 in terms of g and Δt, and the distance h in terms of g and Δt. Then combine these results to eliminate the unknown time interval Δt.

s-5 *(Test problem H-2):* During the time interval Δt, the object's center of mass moves like a particle with a uniformly changing velocity. Therefore, using our result from text section D of Unit 404, the center of mass has a displacement equal to its average (or middle) velocity multiplied by the time interval. Since the center of mass has a final velocity of zero, its middle velocity is just one-half its initial velocity.

s-6 *(Text problem E-8):* The person with larger mass has a larger momentum upon striking the ground, because the velocities of the two persons are the same. Consequently, the change in the momentum of the more massive person is also larger, since the final momentum of both persons is zero.

s-7 *(Text problem H-1):* Apply the relation $d\vec{P}/dt = \vec{F}_{\rm ext}$. In part (a), note that the momentum of the head after the impact is equal in magnitude but *opposite* in direction to the momentum of the head before the impact, so that the change $d\vec{P}$ in the momentum, a vector difference, is not zero. (Verify this for yourself with a diagram.)

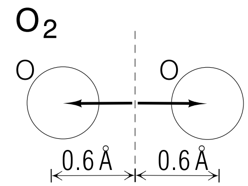
s-8 *(Text problem F-1):* The momentum of a system is only conserved if the total external force on the system is negligible or zero. For example, this is the case for the system of patient and cot because this system is supported vertically and is free to move horizontally with a negligible friction. On the other hand, this is not true of one of the vehicles involved in a collision, even though the vehicle is similarly supported by an icy surface, because this system experiences a horizontal external force due to the other vehicle.

s-9 *(Text problem E-4):* Since any system (which might be part of another system) has a momentum equal to its mass times the velocity of its center of mass, the momenta of the two parts of the patient's body are $m_1\vec{v}_1$ and $m_2\vec{v}_2$. Thus the momentum $\vec{P}$ of the patient's body is $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = 0$. Using this vector relation, you can obtain an equation for $\vec{v}_1$ in terms of the known quantities m_1, m_2, and $\vec{v}_2$.

s-10 *(Text problem B-1):* The internal forces on a system are just the mutual forces between particles in the system. Thus the force on the parachute due to the man and the force on the man due to the parachute are the internal forces. The forces on these particles due to things outside the system, such as the air, are external forces. To find the total external force $F_{\rm ext}$ on the system, first find the value of all external forces on both the man and the parachute, and then find their *vector* sum. (A unit vector or arrow diagram may be helpful.)

s-11 *(Text problem E-2):* Each of the following suggestions should help you evaluate the corresponding statement in text problem E-2: (a) The momentum of a single particle with velocity $\vec{v}$ is just $\vec{P} = m\vec{v}$. Remember that $\vec{v}$ changes if either the particle's speed or direction of motion changes. (b) The momentum of any system is related to the velocity $\vec{V}$ of its center of mass by the equation $\vec{P} = M\vec{V}$, where M is the mass of the system. (c) Compare the momentum of a railroad engine with the momentum of a ping-pong ball having the same velocity! (d) Find the momentum of a system of two particles of equal mass which are traveling in opposite directions with the same speed.

s-12 *(Text problem C-1):* Let us find the position vector $\vec{R}$ of the center of mass of the oxygen molecule. Since there are only two particles in this system, $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$, where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two oxygen atoms, $m_1 = m_2$ is the mass of an oxygen atom, and $M = m_1 + m_2$ is the mass of the system.



If we call the oxygen atom on the left particle 1, the vector $m_1\vec{r}_1 = (16\,\text{amu})(0.6\,\text{Å to the left}) = 9.6\,\text{amu Å to the left}$.

What is the value of $m_2\vec{r}_2$?

► $m_2\vec{r}_2 =$ ____________________

What is the value of the *vector* sum $m_1\vec{r}_1 + m_2\vec{r}_2$?

► $m_1\vec{r}_1 + m_2\vec{r}_2 =$ ____________________

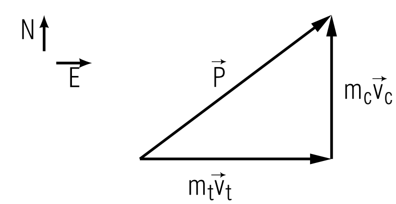
What is the value of the position vector $\vec{R}$ of the center of mass?

► $\vec{R} =$ ____________________

(Answer: 7) Now: Return to text problem C-1.

s-13 *(Text problem E-1):* Part (c): The momentum $\vec{P}$ of the system consisting of both the car and the truck is the *vector* sum $\vec{P} = m_c\vec{v}_c + m_t\vec{v}_t$. From part (a), $m_c\vec{v}_c + (10^3\,\text{kg})(20\,\text{m/s north}) = 2\times10^4\,\text{kg m/s north}$, and from part (b), $m_t\vec{v}_t = 2\times10^4\,\text{kg m/s east}$.

Use a rough vector diagram to construct the vector $\vec{P}$. Be sure to indicate on your diagram that $m_c\vec{v}_c$ and $m_t\vec{v}_t$ have different directions but the same magnitude.



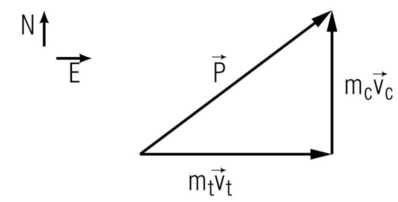
The vectors in this diagram form a right triangle which you can use to find both the magnitude and direction of the momentum $\vec{P}$.

► $\vec{P}$ = ____________________

(Answer: 3) Now: Go to [p-3].

ANSWERS TO PROBLEMS

1. a. 9.6 kg m/s horizontally *away* from the bat
   b. 9.6×10^3 N horizontally away from the bat
   c. $F_{\rm ext}/mg = 6.4 \times 10^3$
2. a. at the rod's center
   b. 15 N upward; 0.10 m/s^2 upward
3.



$\vec{P} = 2.8 \times 10^4$ kg m/s northeast

4. a. $F_{\rm ext} = 2 \times 10^4$ N
   b. 6×10^{-3} meter = 6 mm
5. 1.1×10^4 N for total external force, zero for total internal force
6. a. 1.3×10^{-19} kg m/s $\hat{x}$
   b. 2.2×10^{-26} kg (it is a nitrogen nucleus)
7. 9.6 amu Å to the *right*, zero, zero
8. a. $F_{\rm ext} = mv/dt$ in (1) and (3), $F_{\rm ext} = mv/2dt$ in (2)
   b. (2), (1) and (3) are equally severe
9. 5 kg m/s

101. 10 m/s^2 downward = $\vec{g}$
102. a. $\vec{R} = 0$ for both
   b. midway between the oxygen atoms in O_2, at center of C atom in CO_2
   c. no
103. a. $m_{cd} = 7.3 \times 10^{-26}$ kg

b. $m_{cd} = 2.75 m_0$

c. $m_O = 2.7 \times 10^{-26}$ kg, $m_C = 2.0 \times 10^{-26}$ kg

104. both forces are zero

105. a. Internal: forces exerted on man and parachute by each other. External: remaining forces.

b. zero

c. 100 N downward

106. (a), (c), and (d) are true; (b) is false because two particles with the same velocity but different mass have different momenta

107. zero; no

108. at the geometric center of each; e.g., at the center of the sphere for the bubble and contained air, at the center of page 100 or 101 for the book

109. a. $\vec{P} = m_c \vec{v}_c = 2.0 \times 10^4$ kg m/s north

b. $\vec{P} = m_t \vec{v}_t = 2.0 \times 10^4$ kg m/s east

c. $\vec{P} = m_c \vec{v}_c + m_t \vec{v}_t = 2.8 \times 10^4$ kg m/s northeast

110. $\vec{A} = 1.1\,\mathrm{m/s^2}$ downward

111. (a), (b), (d)

112. a. 1.6 N downward

b. 3.2 kg m/s downward

113. a. 1

b. 6.7×10^4 kg m/s

c. 15 m/s

114. a. 4.0×10^3 kg m/s upward

b. 0.033 second

115. a. $(-36\,\mathrm{kg\,m/s})\hat{x} = 36$ kg m/s away from opponent

b. $(-2.4 \times 10^3\,\mathrm{N})\hat{x} = 2400$ N away from opponent

c. $(-6.0 \times 10^2\,\mathrm{m/s^2})\hat{x} = 600\,\mathrm{m/s^2}$ away from opponent

116. a. $\vec{v}_1 = -m_2 \vec{v}_2 / m_1 = (-0.5\,\mathrm{m/s})\hat{x}$ or 0.5 m/s toward the head

b. Blood: 0.05 meter toward the head. Body: 5×10^{-5} meter toward the feet

117. a. zero

b. 4 m/s

118. a. $\vec{P}$ is horizontal, $d\vec{P}$ vertically downward, and $\vec{P}\prime$ at an angle downward; no

b. no; yes

119. a. 0.5 kg m/s to the right in the figure

b. 500 m/s

120. a. larger than

b. no; the more massive person

121. a. 37°

b. 1.8 or 1.9×10^3 m/s

122. a. $F_{\text{ext}} = M V_0 / \Delta t$

b. $\ell = V_0 \Delta t / 2$

c. $F_{\text{ext}} = m V_0^2 / 2\ell$

d. 2×10^5 N (legs stiff), 2×10^4 N (legs bent); yes, if stiff-legged

123. 9.3 m/s northeast

124. a. 2 m/s south

b. 6×10^5 N south; 6×10^5 N north

125. a. 1.6×10^4 N

b. 8×10^3 N

c. elastic

126. 30 m/s = 67 mph

127. a. $V_0 = \sqrt{2gh}$

b. $F_{\text{ext}} = m(2gh)/2\ell = mgh/\ell$

c. 1.3 meter for the man, 2.0 meter for the woman

128. a. zero

b. $4.0\, m_0$ (it is an alpha particle)

MODEL EXAM

1. **Motion of a hydrogen atom.** A hydrogen atom, consisting of a proton and an electron, is located between oppositely-charged metal plates. The proton is acted on by an electric force of 5×10^{-14} N upward due to the charged plates, and by the electric force of 2×10^{-9} N downward due to the electron. The electron is acted on by an electric force of 5×10^{-14} N downward due to the charged plates, and by an electric force due to the proton. No other forces act on these particles.

   a. What is the total external force on the system consisting of the proton and the electron in this hydrogen atom?

   b. What is the total internal force on this system?

   c. Is the momentum of this system conserved? If not, briefly explain why not.

2. **Impact of a ball with a sidewalk.** A girl throws her 0.1 kg ball vertically downward to see how well it bounces off the sidewalk. Just before it hits the walk, the ball has a velocity of 6 m/s downward. Just after its impact with the walk, the ball has a velocity of 4 m/s upward. We shall assume that during the impact, which lasts 0.01 second, the total external force on the ball is constant.

   a. What is the change in the ball's momentum during the impact?

   b. What is the total external force on the ball during the impact? (Since other forces are negligible, this is the force exerted on the ball by the sidewalk.)

Brief Answers:

1. a. zero or 0 or $\vec{0}$

   b. zero or 0 or $\vec{0}$

   c. If answer (1) is zero, yes

   If answer (1) is not zero, no, because external force not zero

2. a. 1 kg m/s upward

   b. $(1 \times 10^2$ or $100)$ N upward