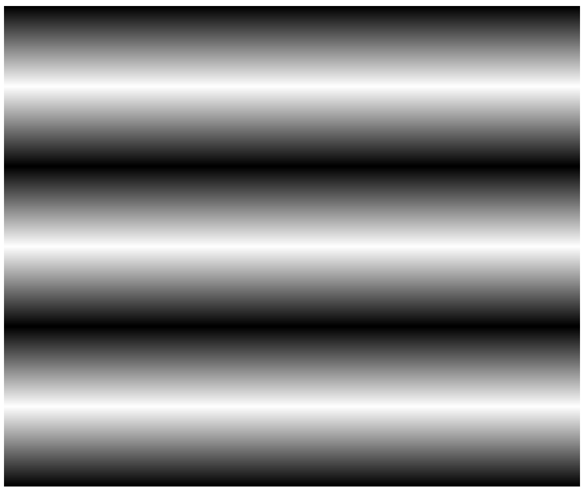
PHYSNET

MISN-0-233

STANDING WAVES IN SHEETS OF MATERIAL



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

STANDING WAVES IN SHEETS OF MATERIAL

by

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Title: **Standing Waves in Sheets Of Material**

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Input Skills:

1. Vocabulary: characteristic (resonant) frequencies, fundamental frequency, harmonics, node, normal modes, overtones, standing wave (MISN-0-232).
2. Write the equation for a one-dimensional traveling wave (MISN-0-201).
3. Add appropriate one-dimensional traveling waves to make a one-dimensional standing wave (MISN-0-232).
4. Given appropriate boundary conditions, calculate the resonant frequencies of a one-dimensional standing wave (MISN-0-232).
5. Calculate the velocity of a transverse wave on a stretched string and of a longitudinal wave in a rod (MISN-0-232).

Output Skills (Knowledge):

K1. Vocabulary: nodal line, two-dimensional standing wave, wave vector.

K2. Very briefly describe the derivation of the two-dimensional differential wave equation.

K3. Add appropriate two-dimensional traveling waves to make a two-dimensional standing wave in a rectangular sheet with fixed edges.

Output Skills (Rule Application):

R1. Given the dimensions of a rectangular sheet with fixed edges, determine its resonance frequencies.

R2. Given the wave vector numbers for a standing wave in a rectangular sheet with fixed edges, determine: (i) the locations of the nodal lines and the frequency. Sketch the nodal lines, peaks, valleys, and directions of the component traveling waves.

External Resources (Optional):

1. *Vibrations and Waves*, A. P. French, W. W. Norton & Co., Inc., NY (1971), p. 185. For availability, see this module's *Local Guide*.

STANDING WAVES IN SHEETS OF MATERIAL

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1. Introduction

Standing waves often occur in one-dimensional systems such as violin strings and organ pipes, but they also can occur in two-dimensional systems such as drumheads, cymbals, bridges and aircraft wings. In these one- and two-dimensional systems each string, pipe, or sheet of material has its own resonant frequencies that depend on the physical shape of the vibrating object, on the material of which it is made, and on its boundary conditions (how it is fastened or not fastened at its ends or edges). The vibrations are desirable in instruments but are extremely undesirable in many engineering structures where they cause fatigue of material and subsequent failure.

The methods used to calculate resonant frequencies are much the same in one- and two-dimensional systems. The circular objects mentioned above, the drumhead and cymbal, are well-known examples of vibrating sheets of material and their resonant frequencies can be readily calculated. However, in this unit we use rectangular instead of circular examples, so that in direct analogy to the one-dimensional case we can write the standing waves as sums of traveling sine waves instead of the less familiar Bessel waves appropriate to a disc-like surface.

2. Standing Waves

2a. Qualitative: Linear Waves. If one sets up a traveling wave in a one-dimensional stretched string, the wave reflects from the ends and thus one soon has traveling waves going in both directions. This means that if there is a wave traveling along a string, there is soon another wave going in the opposite direction. Under the proper conditions, a standing wave results from the interference of two such oppositely-traveling waves.[1]

2b. Qualitative: Sheet Waves. Suppose we set up a two-dimensional traveling wave in a sheet of material, the wave being headed at an angle toward one of the sheet's sides. The wave reflects in turn off each of the sheet's four sides much as billiard balls reflect from billiard table edges. We will wind up with two waves traveling in opposite directions at an angle across the sheet, and two other waves traveling in opposite directions and at a symmetrically reversed angle with respect to the first pair. Four such waves can add to produce standing waves in the sheet of material.

[1]See "Standing Waves in One Dimension" (MISN-0-232).

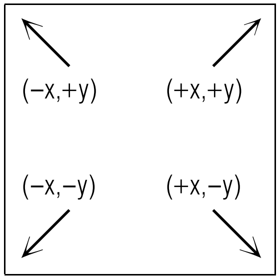


Figure 1. Defining directions used in the text.

2c. Mathematical: Linear Waves. A one-dimensional z-displacement sine wave traveling in the positive x direction can be written as:

$$z = A\sin(\omega t - kx)\,.$$

A similar wave traveling in the negative x-direction is then written:[2]

$$z = A\sin(\omega t + kx)\,.$$

2d. Mathematical: Sheet Waves. In two dimensions, a position on a wave is designated by the vector $\vec{r}$. Similarly, the wave number becomes a "wave vector" $\vec{k}$ in the direction of the wave's propagation. The equation for a single sine wave with frequency ω and wave vector $\vec{k}$ is:

$$z = A\sin(\omega t - \vec{k}\cdot\vec{r})\,.$$

We can use this notation to represent the sum of waves propagating in all four directions in an x-y-plane sheet: in the $(+x,+y)$ direction, the $(-x,-y)$ direction, the $(+x,-y)$ direction, and the $(-x,+y)$ direction (See Fig. 1).

2e. The Standing Wave. A standing wave in a two-dimensional sheet of material must, in general, be a combination of traveling waves in four directions in order to take into account reflections off all four sides of

[2]See "The Wave Equation and It's Solutions" (MISN-0-201).

the sheet. Then any z-displacement standing wave in a sheet can be represented by:

$$z(x,y,t) = A_1 \sin(\omega t - k_x x - k_y y) + A_2 \sin(\omega t + k_x x + k_y y) + A_3 \sin(\omega t - k_x x + k_y y) + A_4 \sin(\omega t + k_x x - k_y y)\,, \qquad (1)$$

where the A's, k's and ω are determined by the dimensions of the sheet and characteristics of the disturbance.

3. Fixing the Resonant Vibrations

3a. Introduction. Just as for the one-dimensional standing wave, we can calculate the relationships among a z-displacement's component amplitudes and then calculate its resonant vibrations. We do this by applying the appropriate boundary conditions—in this case by requiring that $z = 0$ on all four sides where the material is fastened, hence is unable to undergo displacement. For convenience we will orient our coordinate axes so that the sheet edges are at $x = 0$ and $x = a$, and at $y = 0$ and $y = b$ (see Fig. 2).

3b. The X- and Y-Axis Boundaries. Starting first with the edge along the y-axis, we require that $z = 0$ (no displacement) at all points with $x = 0$. Setting $z = 0$ and $x = 0$ in Eq. (1) we obtain:

$$0 = (A_1 + A_4)\sin(\omega t - k_y y) + (A_2 + A_3)\sin(\omega t + k_y y)\,.$$

This must hold for any value of y and any value of t, so the terms with different functions of y and t must vanish separately: $A_1 = -A_4$ and $A_2 = -A_3$. We use these equalities to replace A_3 and A_4 by A_1 and A_2. Taking next the edge along the x-axis, we require that $z = 0$ at all points along the edge along $y = 0$. Making this substitution and collecting similar functions of x and t, we deduce that: $A_1 = A_2 = A$.

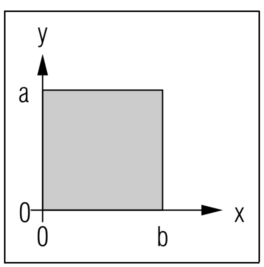


Figure 2. Description of the sheet used as a text illustration.

Our expression for z, Eq. (1), is then:

$$z = A[\sin(\omega t - k_x x - k_y y) + \sin(\omega t + k_x x + k_y y) - \sin(\omega t - k_x x + k_y y) - \sin(\omega t + k_x x - k_y y)]\,. \qquad (2)$$

3c. The Non-Axis Boundaries. A third edge is parallel to the x-axis at $y = b$, and z is zero there also:

$$0 = A\sin[(\omega t - k_x x) - k_y b] + \sin[(\omega t + k_x x) + k_y b] - \sin[(\omega t - k_x x) + k_y b] - \sin[(\omega t + k_x x) - k_y b]\,.$$

Again this must hold for all x and t and so again the terms that contain identical functions of x and t must cancel each other:

$$\sin[(\omega t - k_x x) - k_y b] - \sin[(\omega t - k_x x) + k_y b] = 0\,,$$

and

$$\sin[(\omega t + k_x x) + k_y b] - \sin[(\omega t + k_x x) - k_y b] = 0\,.$$

Now two sine functions will be everywhere equal if and only if their arguments are the same except for an integer multiple of 2π. That is, $\sin c = \sin d$ if and only if: $c + 2\pi n = d$ where n is any integer. In our case,

$$\sin[(\omega t - k_x x) - k_y b] = \sin[(\omega t - k_x x) + k_y b]\,,$$

if and only if:

$$(\omega t - k_x x) - k_y b + 2\pi n_y = (\omega t - k_x x) + k_y\,,$$

where n_y is any integer. The subscript "y" indicates that the integer is associated with boundary conditions in the y variable. Solving the above equation we find:

$$k_y = n_y \pi / b\,. \qquad (3)$$

For all such values of k_y the second pair of sine functions cancels also, and thus, for those particular values of k_y, we have that $z = 0$ at $y = b$ (as required). Just as for standing waves in one dimension, we have found that the y component of the wave vector can have only certain discrete values as long as the sheet of material is kept from vibrating along the edges at $y = 0$ and $y = b$.

The fourth edge is parallel to the y-axis at $x = a$ and since that is clamped also, a calculation for $x = a$ similar to the one for $y = b$ yields the information that k_x can have only certain discrete values too:

$$k_x = n_x \pi / a; \qquad n_x \equiv \text{ any integer}. \tag{4}$$

Thus the boundary conditions allow only a discrete set of values for the propagation vector $\vec{k}$:

$$\vec{k} = (n_x \pi / a)\hat{x} + (n_y \pi / b)\hat{y}\,,$$

where n_x and n_y are restricted to being integers.

4. Nodal Lines, Pictures

4a. Overview. For standing waves on a stretched string one can picture the string vibrating in different ways at different resonant frequencies. The positions of nodes provide a convenient description. For example, at the fundamental frequency the wave has nodes only at the two ends of the string and so the "wavelength" is twice the string length. At the first harmonic there is an additional node in the middle of the string and so the "wavelength" equals the string length. For standing waves in a sheet of material one can also picture the vibrations at various resonant frequencies but now we must specify two numbers for each resonant frequency, n_x and n_y. If we know n_x and n_y, we can calculate the positions of the nodal lines, which are parallel to the axes, and those positions enables us to picture the vibrations.

4b. Example: $n_y = 2$. To illustrate the calculation of nodal lines, we first use the case $n_y = 2$ as an example. Putting this into Eq. (2), Eq. (1) becomes:

$$z = A\left\{\sin\left[(\omega t - k_x x) - \frac{2\pi}{b}y\right] + \sin\left[(\omega t + k_x x) + \frac{2\pi}{b}y\right]\right.$$
$$\left. - \sin\left[(\omega t - k_x x) + \frac{2\pi}{b}y\right] - \sin\left[(\omega t + k_x x) - \frac{2\pi}{b}y\right]\right\}.$$

A nodal line is where $z = 0$ and hence it is at a y value, y_{nl}, where:

$$(\omega t - k_x x) - \frac{2\pi}{b}y_{nl} + 2\pi n = (\omega t - k_x x) + \frac{2\pi}{b}y_{nl}\,.$$

This means that:

$$y_{nl} = nb/2\,.$$

Here n is any integer but of course y can not exceed b, the length of the sheet material. Then $z = 0$ for $y = 0$, $b/2$, and b. We already knew that $z = 0$ at $y = 0$ and b but now we see that for $n_y = 2$ there is an additional nodal line at $y = b/2$.

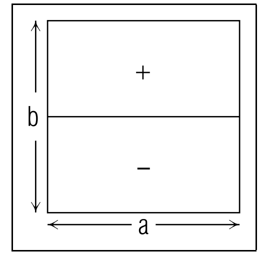


Figure 3. $t = P/4$, mode = (1,2).

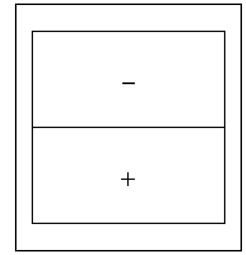


Figure 4. $t = 3P/4$, mode = (1,2).

4c. Example: $n_y = 2$ and $n_x = 1$. Given that $n_y = 2$ produces a nodal line at $y = b/2$, we can set up a complete example by adding the condition $n_x = 1$. To find out where the nodes are in the x-variable, one does a calculation similar to that for the y-variable looking at z as a function of x and putting in $k_x = \pi/a$. This gives $z = 0$ at $x = 0$ and a but at no other values of x; that is, for $n_x = 1$, the only nodal lines are at the edges of the material where the sheet is permanently fixed. To picture what the vibration looks like, we can draw lines on a plane to represent the nodes, and then we know that in between are the antinodes which vibrate up and down as a function of time. For the mode of oscillation with $n_x = 1$ and $n_y = 2$ the appearance of the sheet is as in Figs. 3 and 4 at two fractions of the period of oscillation, P. You should think of those two figures as being two frames in a sequence of five frames: (i) at $t = 0$ the sheet is flat; (ii) at $t = P/4$ the sheet is as shown in Fig. 3; (iii) at $t = P/2$ the sheet is again flat; (iv) at $t = 3P/4$ the sheet is as shown in Fig. 4; (v) at $t = P$ the sheet is again flat.[3]

4d. Other Examples. Figures 5 and 6 show a square sheet at two times for the $(n_x, n_y) = (2, 2)$ mode of vibration, while Figs. 7 and 8 show the same times for the $(n_x, n_y) = (1, 3)$ mode.

5. Frequencies

5a. Overview. In contrast to a one-dimensional system, a material sheet's sequence of resonant frequencies does not in general form a har-

[3] For some beautiful photographs illustrating these modes, see *Vibrations and Waves*, A. P. French, W. W. Norton & Co., Inc., NY (1971), p. 185. See if you can deduce which normal modes are shown there. For availability, see this module's *Local Guide*.

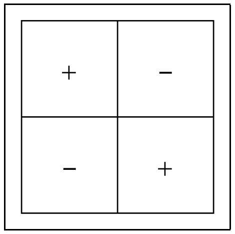


Figure 5. $t = P/4$, mode = (2,2).

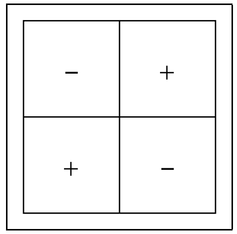


Figure 6. $t = 3P/4$, mode = (2,2).

monic series. That is, the higher resonant frequencies are not generally integral multiples of the fundamental frequency. To calculate the frequency of a two-dimensional wave whose wave vector is known, one must first relate the wave vector to the velocity and then the velocity to the frequency.

5b. Frequency for (n_x,n_y) and Velocity. Interesting properties of standing waves can be deduced by expressing frequency in terms of velocity. Of course a standing wave does not have a velocity, but we can easily determine the velocities of the four traveling waves which make up any particular standing wave in a sheet. For a rectangular sheet of material and a standing wave with wave vector $(\pi n_x/a, \pi n_y/b)$, the wave number of each of the four component traveling waves is:

$$k = \pi\sqrt{(n_x/a)^2 + (n_y/b)^2}\,.$$

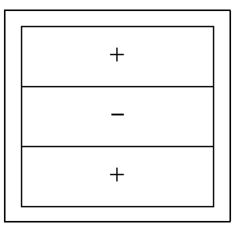


Figure 7. $t = P/4$, mode = (1,3).

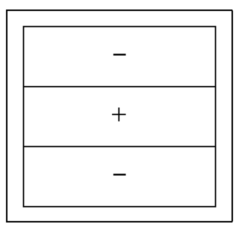


Figure 8. $t = 3P/4$, mode = (1,3).

Recalling that wave number is just the inverse of wavelength times 2π we can relate a component wave's frequency ν to its velocity v:

$$\nu = (v/2)\sqrt{(n_x/a)^2 + (n_y/b)^2}\,.$$

The fundamental (lowest) frequency ν_0 is obtained when $n_x = n_y = 1$. Higher values of n_x and n_y give higher frequencies. Calculate some of them: you will find that sometimes you get an integral multiple of ν_0 but not in general.

5c. Fundamental and Overtones. To find the fundamental and the first few overtones, substitute various small integers for n_x and n_y and calculate the corresponding frequencies. Arrange the frequencies in order of increasing numerical value. The lowest is the fundamental. The next highest is the first overtone, the next highest is the second overtone, etc. Some people prefer to do the calculations on a computer, plugging in various small integers for n_x and n_y.

5c. General Method for Getting the Velocity. The velocity of a wave depends upon the physical properties of the wave's medium and not on the shape or frequency of the wave. Just as for the one dimensional wave, the way to calculate the velocity of a two dimensional wave is to identify the appropriate restoring force and then apply Newton's second law to a small piece of the material to obtain a wave equation.[4] The wave equation now will be two-dimensional, but for an isotropic medium the velocity will be the same in all directions. Once one has the correct form of the wave equation, the velocity is easily identified by comparison to the general form of the wave equation for a sheet in the x-y plane and displacement in the z-direction:

$$v^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^2 z}{\partial t^2}$$

5d. Some Velocities. For a string stretched with tension T and having a mass per unit length ρ_ℓ:

$$v = \sqrt{T/\rho_\ell}\,.$$

For a pliable sheet stretched with tension per unit length T_ℓ and having a mass per unit area ρ_A, the stretched string derivation can be modified to

[4] Waves are created because, when the medium is disturbed, the restoring force acts to return the medium to its equilibrium position. The mass of the medium causes it to "overshoot" the equilibrium position while the strain in the medium causes the disturbance to spread, to "propagate."

apply Newton's second law to an element of area $dA = dx\,dy$. Adding the x- and y-direction forces on the element, one obtains the two dimensional wave equation with:

$$v = \sqrt{T_\ell/\rho_A}\,.$$

For a stiff sheet with shear modulus G and mass per unit volume ρ_v, a similar derivation gives:

$$v = \sqrt{G/\rho_v}\,.$$

Such formulas can be found in handbooks, along with values for ρ and G for various materials.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

LOCAL GUIDE

The book listed in this module's *ID Sheet* is on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Tell the person at the desk that you want a book that is on reserve for CBI (a BOOK, not a reading). Then tell the person the name of the book you want.

PROBLEM SUPPLEMENT

Note: Problems 9-11 also occur in this module's *Model Exam.*

1. Suppose a sheet of material is disturbed in such a way as to set up two traveling sine waves, one going in the $(+x,+y)$ direction and one in the $(+x,-y)$ direction. Their amplitudes and wavelengths are equal and the wave displacement values at the origin are zero at time zero. Before reflection from the boundary takes place, determine these properties of the resultant wave:

   a. Its mathematical form.

   b. Its direction.

   c. Its wave number and wavelength in terms of the wave number and wavelength of its components.

   d. Its nodal lines in terms of the wavelength of its components.

2. Suppose you have a square drumhead of side a and traveling wave velocity v. What is its fundamental frequency? What are the next six higher frequencies, expressed in units of the fundamental frequency?

3. Given that the equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\rho}{G}\frac{\partial^2}{\partial t^2}\right] z(x,y,t) = 0$$

   is obeyed for transverse waves in a particular metallic sheet on the surface of an airplane wing. If the above sheet is struck a sharp transverse blow, with what speed will transverse waves travel away from the point of impact? If you knew the composition of the metal, where would you expect to find values for G and ρ?

4. Given that the metallic sheet is rectangular and rigidly fixed along its edges but is otherwise unsupported. Show that the function

$$\begin{aligned} z =& A[\sin(\omega t - k_x x - k_y y) + \sin(\omega t + k_x x + k_y y) \\ &- \sin(\omega t - k_x x + k_y y) - \sin(\omega t + k_x x - k_y y)]\,. \end{aligned}$$

   not only satisfies the equation in Problem (1) (BE BRIEF) but also gives a correct description of the non-motion of those two edges of the sheet which are along the lines $x = 0$ and $y = 0$.

5. Derive the resonant frequencies by forcing z to give a correct description of the non-motion of the other two sheet edges, which are located along the lines $x = -b$ and $y = -a$, $b \neq a$. The answer for $b = a$ is:

$$\nu(n_x, n_y) = (v/2a)\sqrt{n_x^2 + n_y^2}$$

   where n_x, n_y are integers.

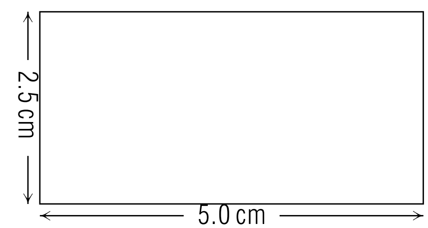
6. From Problems (4) and (5), derive the nodal lines for the resonance $n_x = 2$, $n_y = 3$, $b \neq a$.

7. Sketch the sheet at $t = P/4$, indicating the peaks, valleys, and nodal lines, for the case in Problem (6). If you wish to check your sketch, you can solve for $z(-b/4, y, P/4)$ and $\xi(-3b/4, y, P/4)$. You may wish to use:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B\,.$$

8. Suppose it takes N flexures for the sheet to fatigue sufficiently to fail. If there is a sizable component of the sheet's fundamental frequency in the exhaust noise, find the amount of flight time to failure.

9. Given a pliable rectangular sheet clamped on its entire perimeter, with the dimensions given below, under a surface tension T_ℓ of 13.7 N/m and with mass per unit area of 1.92 kg/m^2, calculate the five lowest normal mode frequencies.



10. For the above system of problem 3, determine the location of the nodal lines of the five normal modes and sketch the appearance of the sheet for each normal mode, identifying the peaks and valleys at one extremum of motion.

11. Assuming the 5.0 cm dimension to be the x-direction and that normal modes are labeled by the ordered pair (n_x,n_y), calculate the resonant frequency of the (7,9) mode.

Brief Answers:

1. a. $z = A[\sin(\omega t - kx - ky) + \sin(\omega t - kx + ky)]$

   b. Parallel to the x-axis.

   c. $k_{\text{resultant}} = \sqrt{2}k$; $\lambda_{\text{resultant}} = \lambda\sqrt{2}$

   d. At $y = n\lambda/2$, where n is any integer.

2. $\nu_0 = (v/2a)\sqrt{2}$; $\nu_1/\nu_0 \ldots \nu_6/\nu_0 = \sqrt{2.5}$, $\sqrt{4.5}$, $\sqrt{6.5}$, $\sqrt{8.5}$, $\sqrt{9}$

3. $v = \sqrt{G/\rho}$. *Handbook of Chemistry and Physics*, for example.

4. For any of the four terms, $\partial^2/\partial x^2(\text{term}) = (-k_x^2)(\text{term})$, etc.

   So: (sum of operators) $\times$ (term)$= (-k_x^2 - k_y^2 + \rho\omega^2/G)\times$ (term)

   but: $(-k_x^2 - k_y^2 + \rho\omega^2/G) = (-k^2 + k^2) = 0$. Check.

   The two substitutions and cancellations can be seen by inspection.

5. $\nu = (v/2)\sqrt{(n_x/b)^2 + (n_y/a)^2}$

6. $x = -b/2$; $y = -a/3$; $y = -2a/3$

7. $z(-b/4, y, P/4) = 4A\sin(3\pi y/a)$,

   $z(-3b/4, y, P/4) = -4A\sin(3\pi y/a)$

8. $T = \dfrac{N}{2\nu(n_x = 1, n_y = 1)}$.

9. Fundamental: $\nu_0 = 59.7\,\text{Hz}$

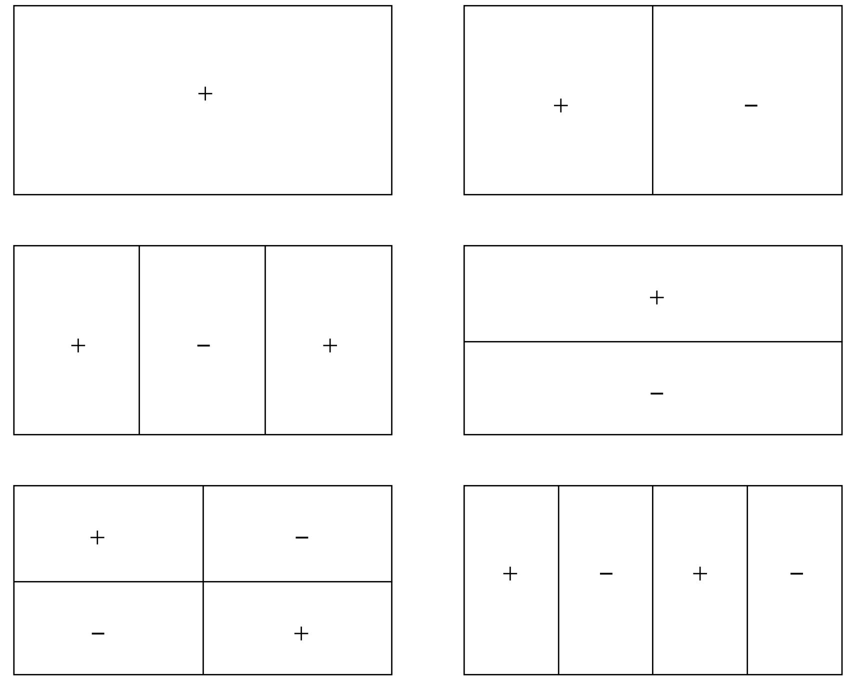
   First Overtone: $\nu_1 = 75.5\,\text{Hz}$

   Second Overtone: $\nu_2 = 96.3\,\text{Hz}$

   Third Overtone: $\nu_3 = 110.1\,\text{Hz}$

   Fourth Overtone: $\nu_4 = 119.5\,\text{Hz}$

10.

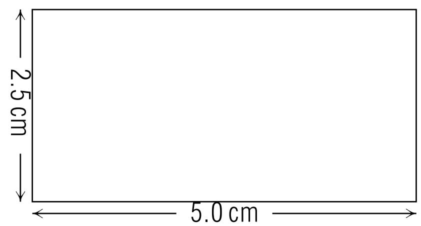


11. $\nu(7,9) = 515.9\,\text{Hz}$

MODEL EXAM

1. See Output Skills K1-K3.

2. Given a pliable rectangular sheet clamped on its entire perimeter, with the dimensions given below, under a surface tension T_ℓ of 13.7 N/m and with mass per unit area of 1.92 kg/m^2, calculate the five lowest normal mode frequencies.



3. For the above system of problem 2, determine the location of the nodal lines of the five normal modes and sketch the appearance of the sheet for each normal mode, identifying the peaks and valleys at one extremum of motion.

4. Assuming the 5.0 cm dimension to be the x-direction and that normal modes are labeled by the ordered pair (n_x,n_y), calculate the resonant frequency of the (7,9) mode.

Brief Answers:

1. See this module's *text*.

2. See Problem 9 in this module's *Problem Supplement*.

3. See Problem 10 in this module's *Problem Supplement*.

4. See Problem 11 in this module's *Problem Supplement*.