PHYSNET

MISN-0-1

SIMPLE DIFFERENTIATION AND INTEGRATION

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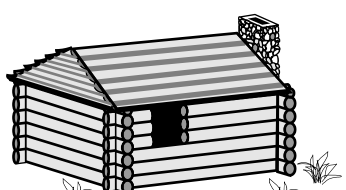

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SIMPLE DIFFERENTIATION AND INTEGRATION

by

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Input Skills:

1. Vocabulary: limit, trigonometric identity, sin, cos, tan, inverse trigonometric function, curve tangent, function graph, graphical slope, exp, log, ln (MISN-0-401).

K2. Define "rate," state two of its general properties and give the rates for $x = C$, $x = Ct$, $x = Ct^2$ (MISN-0-404).

K3. Explain what is meant by "small enough" and how it is used to distinguish a difference from a derivative (MISN-0-404).

K4. Define "velocity," for motion in a straight line, in these cases: (a) v is constant; (b) v changes uniformly; (c) v is any function (MISN-0-404).

Output Skills (Rule Application):

R1. Differentiate polynomial, exponential, logarithmic, and trigonometric functions.

R2. Locate the maxima and minima of any given function.

R3. Solve indefinite integrals of polynomial functions.

R4. Evaluate definite integrals of polynomial functions.

R5. Determine the area under the curve of a given function using the definite integral.

SIMPLE DIFFERENTIATION AND INTEGRATION

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J. S. Kovacs, Michigan State University

1. Introduction

The description of physical phenomena without the results of measurements and without mathematics is, at best, incomplete and very limited in its scope. For example, the observation that the moon orbits the earth is a qualitative description of that motion. The description becomes quantitative when measurements are made that give the position of the moon relative to the earth for various times. With the accumulation of such data, the description begins to enter the realm of science when, on the basis of these data, the location of the moon can be predicted for times in the future. However, the description achieves full scientific status when it can also be arrived at from a basic principle or physical law which not only fully predicts all aspects of the observed motion of the moon around the earth, but also can be applied to problems of the motion of other objects in other environments. Relating various observations through physical laws invariably involves the use of mathematics. In this module, we will consider some of the basic mathematical skills necessary for understanding such applications.

The functions whose derivatives are dealt with here constitute most of the kinds of functions encountered in an introductory physics course. For ready reference, we tabulate the rules and frequently used derivatives in Appendix B.

2. Differentiating Common Functions

2a. Definition of the Derivative. The derivative of a function of a single independent variable, $y(x)$, with respect to that independent variable, x, is defined as:

$$\frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \tag{1}$$

where Δx is a small incremental change in the value of x that is required to go to zero as the prescribed ratio is examined.

As an example, consider where a is a constant. This relationship defines a value for y for each value of x except at $x = 0$. The derivative of this function is:

$$\begin{aligned} \frac{dy(x)}{dx} &= \frac{d}{dx}\left(\frac{a}{x}\right) \\ &= \lim_{\Delta x \to 0} \frac{\dfrac{a}{x+\Delta x} - \dfrac{a}{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-a}{x(x+\Delta x)} \frac{\Delta x}{\Delta x} \\ &= -\frac{a}{x^2}. \end{aligned}$$

This function defines a value of the derivative of y for each value of x except at $x = 0$.

2b. Derivatives of Simple Algebraic Functions. It is not necessary to go through the cumbersome limiting procedure each time a derivative of a function needs to be determined. Instead, some general rules can be used. For example, if the function is some power of x, $y(x) = ax^p$, where p is any positive or negative number, then:

$$\frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{a(x+\Delta x)^p - ax^p}{\Delta x}$$

becomes, after applying the binomial theorem and the limiting procedure, *Help: [S-2]*[1]

$$\frac{d(ax^p)}{dx} = pax^{p-1}. \tag{2}$$

Equation (2) can be applied any time the derivative of any power of the independent variable is desired. Because the derivative of a sum of terms is the sum of the derivatives of the terms, the above result can be used to produce the derivative of any polynomial.[2] For example, if

$$y(t) = y_0 + v_0 t - \frac{gt^2}{2}$$

[1] The [S-2] means that, if you need help, turn to the Special Assistance Supplement at the end of this module and look at Sequence [S-2].

[2] A polynomial is a sum of integer powers of the independent variable with constant coefficients. A polynomial of degree n in general is written as:

$$y(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n. \qquad (a_0 \neq 0)$$

where y_0, v_0 and g are constants, and t is the independent variable, then Eq. (2) produces:

$$\frac{dy(t)}{dt} = v_0 - gt.$$

2c. Second Derivatives. The derivative of $dy(t)/dt$, called the second derivative of $y(t)$, is written:

$$\frac{d^2y(t)}{dt^2}.$$

For example, for the function $y(t) = y_0 + v_0 t - gt^2/2$, double application of rule (2) produces:

$$\frac{d}{dt}\left(\frac{dy(t)}{dt}\right) = -g.$$

This $y(t)$ is a polynomial of degree two, its first derivative is a polynomial of degree one and its second derivative is a polynomial of degree zero (just a constant).

2d. Derivative of a Product. The derivatives of more complicated algebraic functions, such as the products or ratios of polynomials, can also be readily found with the aid of some basic rules. Consider a function which can be written as the product (gf) of two functions of the same independent variable; g(x) and f(x). The derivative of this function, according to the definition, is:

$$\frac{d(gf)}{dx} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)f(x+\Delta x) - g(x)f(x)}{\Delta x}.$$

If $g(x)f(x+\Delta x)$ is added and subtracted in the numerator, the expression may be rewritten in this way:

$$\frac{d(gf)}{dx} = \lim_{\Delta x \to 0} \left(g(x)h_1(x) + h_2(x)f(x+\Delta x)\right),$$

where

$$h_1(x) \equiv \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

and

$$h_2(x) \equiv \frac{g(x+\Delta x) - g(x)}{\Delta x}.$$

In the limit that Δx goes to zero, the two quantities $h_1(x)$ and $h_2(x)$ go to the derivatives of the respective functions while the $f(x+\Delta x)$ multiplying $h_2(x)$ goes simply to $f(x)$. Thus:

$$\frac{d(gf)}{dx} = \left(\frac{dg}{dx}\right) f + g\left(\frac{df}{dx}\right). \tag{3}$$

Similarly:

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \frac{\left(\frac{dg}{dx}\right) f - g\left(\frac{df}{dx}\right)}{f^2}. \tag{4}$$

These rules enable us to find the derivative of rational algebraic functions (polynomials or ratios of polynomials), but not of irrational algebraic functions (such as square roots of polynomials).

2e. The Chain Rule. To enable us to find the derivative of an irrational algebraic function, we need to use the "chain rule."

We are given f as a function of x and we suppose we can find its derivative,

$$\frac{df}{dx}. \tag{5}$$

Now suppose that x itself is a function of t so that f is also a function of t. Then the chain rule says that the derivative of $f(x)$ with respect to t is:

$$\frac{df(x(t))}{dt} = \left(\frac{df}{dx}\right)\left(\frac{dx}{dt}\right). \tag{6}$$

Using this, the derivative of a function such as

$$F(x) = (ax^2 + bx + c)^{1/2},$$

where a, b, and c are constants, can readily be shown to be: *Help: [S-4]*

$$\frac{dF(x)}{dx} = \frac{2ax + b}{2(ax^2 + bx + c)^{1/2}}. \tag{7}$$

2f. Derivatives of Trigonometric Functions. We can evaluate the derivatives of transcendental functions, such as $\sin x$, which cannot be expressed as rational or irrational algebraic functions, using rules developed from the definition of the derivative. Consider the function $y(x) = \sin(kx + \delta)$ where k and δ are constants. The derivative of $y(x)$ is:

$$\frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{\sin(kx + k\Delta x + \delta) - \sin(kx + \delta)}{\Delta x}.$$

Using a trigonometric identity,[3]

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) cos\left(\frac{A+B}{2}\right),$$

and the limit

$$\lim_{\theta\to 0}\frac{sin\theta}{\theta} = 1,$$

we find that: *Help: [S-3]*

$$\frac{d}{dx}[\sin(kx+\delta)] = k\cos(kx+\delta). \quad (8)$$

The derivative of $\cos(kx+\delta)$ can be obtained directly from the above result using the relations

$$\cos(kx+\delta) = -\sin\left(kx+\delta+\frac{\pi}{2}\right)$$

and

$$\cos\left(kx+\delta-\frac{\pi}{2}\right) = sin(kx+\delta), \quad (9)$$

yielding:

$$\frac{d}{dx}\cos(kx+\delta) = -k\sin(kx+\delta). \quad (10)$$

The derivatives of other trigonometric functions may be obtained from the above results and the appropriate trigonometric identities. For example, the derivative of $\tan\theta$ with respect to θ is:

$$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{1}{\cos^2\theta} = \sec^2\theta. \quad (11)$$

Similarly, the derivatives of the inverse trigonometric functions, such as $\theta = \arctan x$ (also written as $\tan^{-1} x$, which means θ is the angle whose tangent is x) can also be obtained from the above results using the rule:

$$\frac{dy(x)}{dx} = \frac{1}{\frac{dx(y)}{dy}}; \left(\frac{dx}{dy} \neq 0\right). \quad (12)$$

Frequently used derivatives of trigonometric functions are listed in Appendix B. /par

[3]See *Handbook of Chemistry and Physics*, Chemical Rubber Publishing Co. The identity can be proved by recalling the expansion for $\sin(x \pm y)$ and using the substitutions $A = x + y$ and $B = x - y$.

2g. Derivatives of Exponentials and Logarithms. None of our rules, developed from algebraic and trigonometric functions, apply to exponential and logarithmic functions, so here it's necessary to begin again with the definition of the derivative. If $x = \log_a F$, then the derivative of x with respect to F is:

$$\begin{aligned}\frac{dx}{dF} &= \frac{d}{dF}(\log_a F)\\ &= \lim_{\Delta F\to 0}\frac{\log_a(F+\Delta F) - \log_a(F)}{\Delta F}\\ &= \lim_{\Delta F\to 0}\frac{\log_a\left(1+\frac{\Delta F}{F}\right)}{\Delta F}\\ &= \lim_{\Delta F\to 0}\frac{\frac{\Delta F}{F}\log_a\left[\left(1+\frac{\Delta F}{F}\right)^{F/\Delta F}\right]}{\Delta F}\end{aligned}$$

Here we have used property (3) of logarithms (see Appendix A). What is the limit of the argument of this logarithm?[4] That is, what is:

$$y = \lim_{x\to 0}(1+x)^{1/x}?$$

This limit has a definite value, as can be seen by evaluating it and plotting it for a few values of x on both sides of zero. The value of the limit is the transcendental number e, which to 9 decimal places is

$$e = 2.718281828\ldots$$

As we shall see, e's actual value is often of no real interest in physical problems, but it enters in a natural way as the exponential base appropriate to the mathematical description of many different kinds of physical phenomena. /par

Completing the development of the derivative of the logarithm, we get

$$\frac{d(\log_a F)}{dF} = \frac{\log_a e}{F} \quad (13)$$

If we choose the base $a = e$, then the constant $\log_a e$ becomes 1 (see property (4) of logarithms, Appendix A). Because this makes life simpler,

[4]The limit of the logarithm of a function is the same as the logarithm of the limit of the same function.

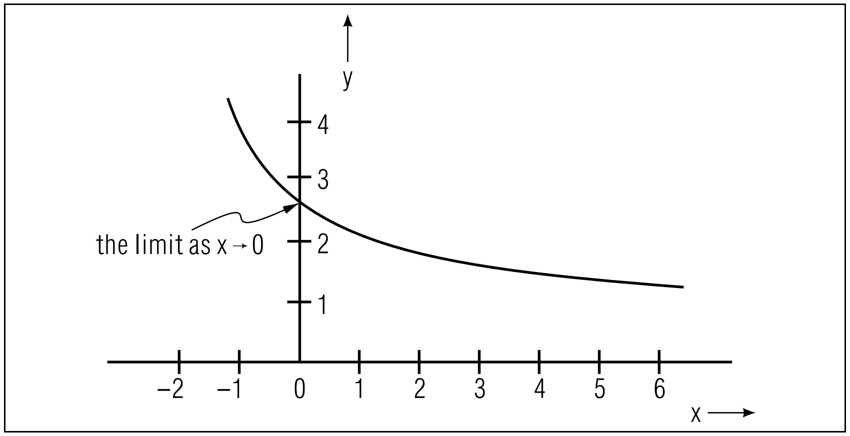

Figure 1. A graph of $(1+x)^{1/x}$ in the vicinity of $x=0$.

we will restrict the remainder of our review of derivatives of logarithms and exponentials to that base. Furthermore, we write "$\log_e$" as simply "ℓn." Thus, for a variable F, the derivative of "$\log F$ to the base e," $\ell n\, F$, is:

$$\frac{d(\ell n\, F)}{dF} = \frac{1}{F}. \qquad (14)$$

We can obtain the derivative of e^x with respect to x from the reciprocal relation between the derivatives when the roles of the dependent and independent variables are interchanged [See Eq. (11)]. We use

$$\begin{aligned} x(F) &= \ell n\, F \\ F(x) &= e^x \\ \frac{dx(F)}{dF} &= \frac{1}{dF(x)/dx} \end{aligned}$$

to get:

$$\frac{d}{dx}\left(e^x\right) = F = e^x. \qquad (15)$$

Thus the exponential (to the base e) is its own derivative! Finally, using the chain rule, it can be shown that:

$$\frac{d}{dx}\left(e^{mx}\right) = me^{mx}. \qquad (16)$$

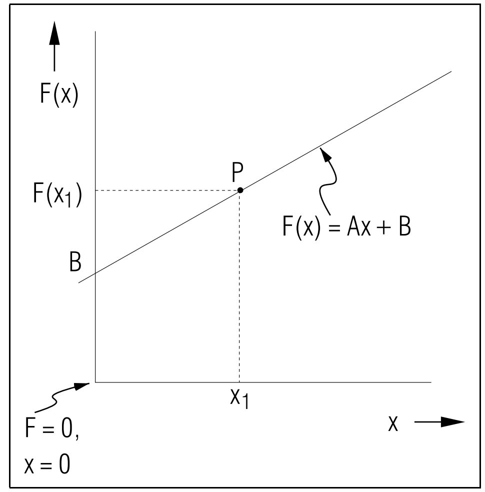

Figure 2. The graph of $F(x) = Ax + b$.

3. Derivatives as Graphical Slopes

3a. Graph of An Equation. The functional relationship between two variables, often expressed in the form of an equation, may also be displayed on a graph. The graphical representation has distinct visual advantages, presenting all of the properties of the function in one picture. All of the points that lie on the curve are points which satisfy the equation connecting the variables.

As an example, consider the functional relationship between F and x given by the equation

$$F(x) = Ax + B,$$

where A and B are constants. For every value of x there is one value $F(x)$ and for every value of $F(x)$ there is one value of x. The points for which this correspondence exists can be connected by a continuous curve. This graph of the equation can be plotted on a rectangular coordinate system in which $F(x)$ and x are the perpendicular axes (See Fig. 2.). The graph of this equation is a straight line, reflecting the linear relationship between F and x. Given a value $x = x_1$, the value of $F(x)$ which satisfies the equation can be obtained from the equation $F(x_1) = Ax_1 + B$ or can be determined from the graph [point P on the graph in Fig. 2 has coordinates x_1, $F(x_1)$].[5]

[5]On a two-dimensional rectangular plot it is customary to represent points by the ordered pair of numbers $[x, y]$ which are, respectively, the horizontal and vertical co-

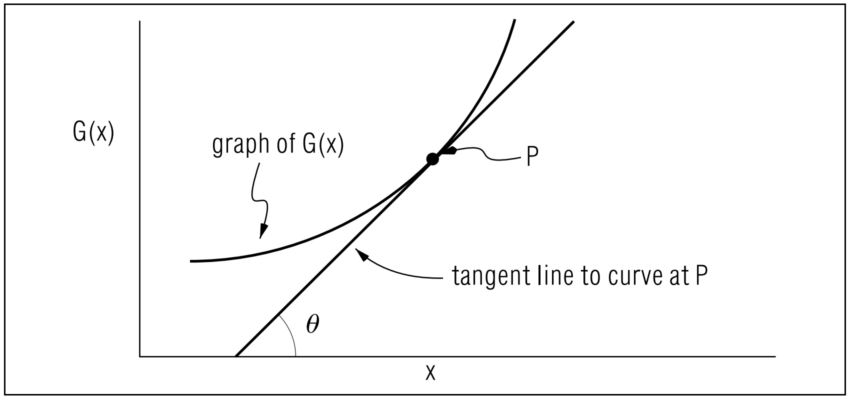

Figure 3. The slope of a curve $G(x)$, at some point P, equals the slope of the tangent to the curve at that point.

3b. Slope of a Straight Line. A property associated with equations which is of importance in physical applications and which can most easily be discussed with reference to the graph of the equation is the rate at which one variable changes due to changes in the other. For a graph which is a straight line, the rate is the same as the slope of the line (the slope being the tangent of the angle that the line makes with the horizontal axis). Referring to Fig. 2, if P_1 and P_2 are two points whose coordinates are $[x_1,\ F(x_1)]$ and $[x_2,\ F(x_2)]$, then the rate at which F changes between those two points relative to the corresponding change in x is:

$$\frac{\Delta F}{\Delta x} = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

Let $\Delta x = x_2 - x_1$ so that $F(x_2) = F(x_1 + \Delta x)$ and the slope of the line may be written as:

$$\text{slope} \equiv \frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} \qquad \text{(straight line)}. \tag{17}$$

For a straight line, the result of Eq. (17) is clearly the same no matter what size we pick for Δx. That will not be the case for curves that are not straight lines.

3c. Slopes of Curves. If we take the limit of Eq. (17) as $\Delta x \to 0$, this expression for the slope becomes the derivative of F with respect to

ordinates of the point.

x, and we can define the slope at a single point on any curve as:

$$\text{slope at point } x_1 \equiv \left.\frac{dF(x)}{dx}\right|_{x=x_1}. \tag{18}$$

For a function whose graph is not a straight line, the slope does depend upon the point at which it is evaluated. The slope in such cases is the slope of the straight line that is tangent to the curve at the point in question (see Fig. 3).

3d. Finding Maximum and Minimum Points. The collection of all maximum and minimum points of a function can be found by finding the points of zero slope (see Fig. 4). From the definition of slope [see Eq. (18)] it is apparent that the slopes at P_1 and P_5 in Fig. 4 are positive. However, the slope at P_3, on the decreasing part of the curve, is negative. At point P_2, where the tangent to the curve is horizontal, the slope is zero. Similarly, the slope is zero at P_4. The two points, P_2 and P_4, are, respectively, a “maximum point” and a “minimum point” of the curve. At P_2 all points on the curve in the immediate neighborhood of P_2 have $y(x)$ less than the corresponding value at P_2, while at P_4 all points in the neighborhood of P_4 have $y(x)$ greater than the value of P_4. This illustration indicates the method for determining the maximum and minimum points of a function: Find the values of x which make the derivative of the function equal to zero.

▷ Show that the locations of the maximum and minima of the function

$$y(x) = 5x^3 - 2x^2 - 3x + 2.$$

are at: $x = 3/5; x = -1/3$. *Help: [S-1]* Show that the first is where y is a minimum, the second where it is a maximum. *Help: [S-1]*

3e. Separating into Maxima and Minima. How do you determine whether a point of zero slope is a maximum or a minimum? You could plot a graph and examine it, or you could evaluate the second derivative of the function at the point in question. Referring to Fig. 4, the slope of the curve is positive to the left of P_2, zero at P_2, and negative to the right of P_2. Hence the slope is decreasing. The derivative of the slope, the second derivative of the function $y(x)$ itself, is therefore negative. Thus the second derivative is negative at a maximum—this is because the second derivative is the “bending function” and at a maximum it is bending the function down, negatively. Correspondingly, at the location of a minimum, the second derivative of the function, the “bending,” is positive.

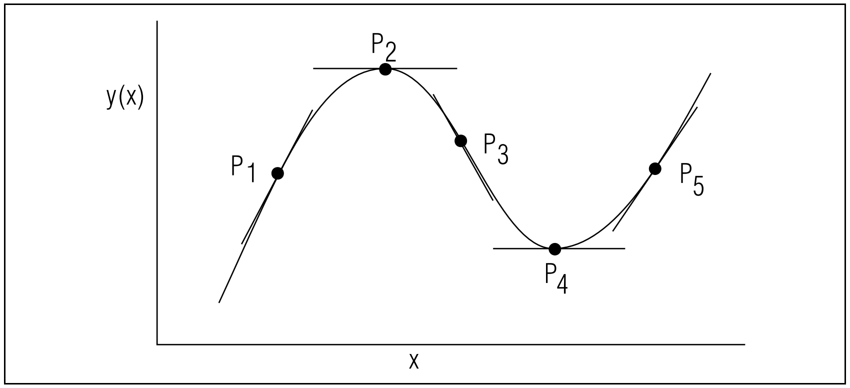

Figure 4. For the curve shown, the slopes at P_1 and P_5 are positive numbers while the slope at P_3 is negative. At the points P_2 and P_4, where the tangent line is horizontal, the slope is zero.

▷ See for yourself that the function

$$y(x) = 2x^3 - 3x^2 + 4,$$

has a maximum at point (0,4) and a minimum at point (1,3).

4. Indefinite and Definite Integrals

4a. Indefinite Integrals. Knowing the derivative of a function with respect to an independent variable, we often wish to determine the function itself. The inverse of the derivative is what is needed, and the procedure is called "integration." The result of this procedure, the inverse of the derivative, is called "the integral" of the function. For example, because $2x$ is the derivative of x^2 with respect to x, the integral of $2x$ with respect to x is x^2. However, $x^2 + 4$ is also the integral of $2x$, as is $x^2 - 10$. In fact, because the derivative eliminates any additive constant, the integral of $2x$ is $x^2 + C$, where C is an unknown constant. Because of the presence of the indefinite constant, $x^2 + C$ is called the "indefinite integral" of $2x$. This process of reversing differentiation, the indefinite integral, is customarily written in the form:

$$\int \frac{df(x)}{dx}\, dx = f(x) + C.$$

That is, the indefinite integral of the derivative of some function $y(x)$ is just the function itself plus a constant. In general, an indefinite integral

is written as:

$$\int y(x)\, dx$$

and it is up to the user to determine what function $y(x)$ is a derivative of, using either standard techniques or a symbolic computer program. The function being integrated, $y(x)$, is called "the integrand of the integral." Examples:

$$(i) \quad \int \left(4x^2 + 7\right) dx = \frac{4}{3}x^3 + 7x + C$$

$$(ii) \quad \int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

$$(iii) \quad \int \frac{dI(x)}{dx} dx = I(x) + C \qquad \text{for any } I(x).$$

The last result, (iii), can be verified by taking the derivative with respect to x of the right side of each equation, thereby producing the integrand of the integral on the left. The integrals of other frequently encountered functions, such as those appearing to the right of the equality sign for entries in Appendix B, may be determined by identifying the integrands as the derivative of sought-for answers as illustrated in (iii).

4b. Other Integration Techniques. Tables of integrals of functions of many kinds are available and are the quickest technique for evaluating the integrals (anti-derivatives) of whatever integrand may be at hand.[6] There are also computer programs that can find the anti-derivative for you.[7]

4c. The Definite Integral as a Change in Quantity. Another use of the integral is to use a known rate of change of a quantity, given as a function of time, to find the change in the total quantity between two times. For example, we may know the electric current in a circuit as a function of time and we wish to know how much charge accumulated on a capacitor in the circuit over a certain time interval. We may know the flow rate of a chemical in a pipe as a function of time and wish to know how

[6]For example, *A Short Table of Integrals*, Third Revised Edition, B. O. Pierce, Ginn and Co. (1929). There is also a table of integrals included in the *Handbook of Chemistry and Physics*, Chemical Rubber Publishing Co., any edition.

[7]See *Mathematica*, S. Wolfram, Addison-Wesley Publ. Co. (1991), `http://www.wolfram.com`, and *Computing with MAPLE*, Francis Wright, Chapman & Hall/CRC (2001) `http://www.maplesoft.com`. There are some 400 books on MAPLE and at least as many on MATHEMATICA.

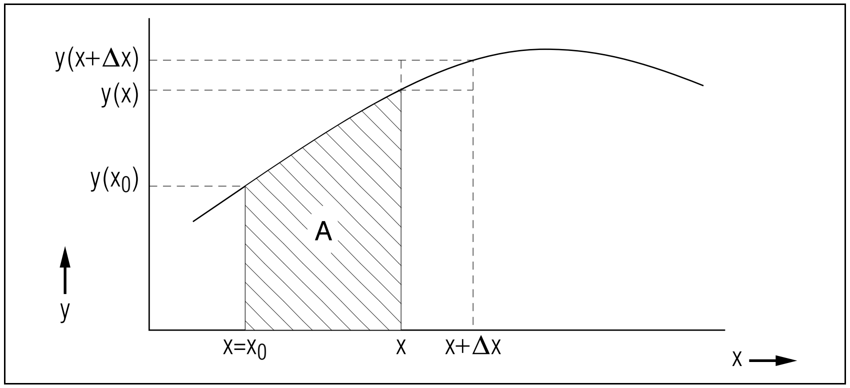

Figure 5. The definite integral of $y(x)$ from x_0 to x is the area A that is under the curve and bounded by x_0 and x.

much of the chemical was delivered by the pipe from one time to another. We may know the speedometer reading of a vehicle as a function of time and we wish to know how far the vehicle traveled between two times. In all three cases we can say we have the rate of a quantity, say $dQ(t)/dt$, as a function of time and we wish to find $\Delta Q \equiv Q(t_2) - Q(t_1)$, the change in total Q from time t_1 to time t_2.[8] If the rate is constant then ΔQ is just the rate times the time interval:

$$\Delta Q = \frac{dQ(t)}{dt}(t_2 - t_1)\,, \qquad \text{(constant rate)}$$

and the apparatus of integration is not needed. However, if the rate is not constant then there is no single rate and one cannot take "rate times time." If the rate is, say, $dQ/dt = at^2 + bt^3$, then we can take the anti-derivative and get the amount as a function of time: $Q(t) = (a/3)t^3 + (b/4)t^4 + C$. The amount that Q changed from time t_1 to time t_2 is then the amount at time t_2 minus the amount at time t_1: $Q(t_2) - Q(t_1) = (a/3)(t_2^3 - t_1^3) + (b/4)(t_2^4 - t_1^4)$. Note that the unknown constant C is gone and the answer is definite. The result is called the "definite integral" and is written with the two times as the "lower limit" and the "upper limit" of the integral:

$$\Delta Q = Q(t_2) - Q(t_1) \equiv \int_{t_1}^{t_2} \frac{dQ(t)}{dt}\,dt\,.$$

[8] A change in any quantity is commonly written using the upper case Greek letter Δ.

We call the example "the integral from t_1 to t_2 of $(at^2 + bt^3)$" and we write it as:

$$\Delta Q = \int_{t_1}^{t_2} (at^2 + bt^3)\,dt\,.$$

We write the *result* of the integration this way:

$$\int_{t_1}^{t_2} (at^2 + bt^3)\,dt = \left|(a/3)t^3 + (b/4)t^4\right|_{t_1}^{t_2}\,,$$

where the vertical bars means that one is to evaluate the expression between them at the indicated upper limit minus the same expression evaluated at the indicated lower limit. In general,

$$|f(x)|_a^b \equiv f(b) - f(a)\,.$$

The first vertical bar is sometimes omitted if that causes no ambiguity. Note that vertical bars without limits indicate "absolute value," a totally unrelated concept. Then for our example:

$$\int_{t_1}^{t_2} (at^2 + bt^3)\,dt = (a/3)(t_2^3 - t_1^3) + (b/4)(t_2^4 - t_1^4)\,.$$

▷ Show that:

$$\int_{1\,\mathrm{s}}^{2\,\mathrm{s}} (4\,\mathrm{m/s^4}t^3 + 5\,\mathrm{m/s^5}t^4)\,dt = 46\,\mathrm{m}\,.$$

4d. The Definite Integral as an Area. The definite integral of a function between a lower and an upper limit equals the area under the function's curve between those two limits (see Fig. 5). For proof, consider the graph of some function $y(x)$ shown in Fig. 5. The area under the curve, between the value x_0 and some other value x, is the area A shown in the figure. If x is increased by the incremental amount Δx, the area under the curve increases by some amount we label ΔA. This amount ΔA is greater than the area of the rectangle of area $y(x)\,\Delta x$ (the smaller dotted rectangle to the right of A) and less than the area of the larger rectangle $y(x + \Delta x)\Delta x$. The incremental area may be written as:

$$\Delta A = y(x)\,\Delta x + \text{ an amount less than } [y(x + \Delta x) - y(x)]\,\Delta x.$$

Then:

$$\frac{\Delta A}{\Delta x} = y(x) + \text{ an amount less than } y(x + \Delta x) - y(x).$$

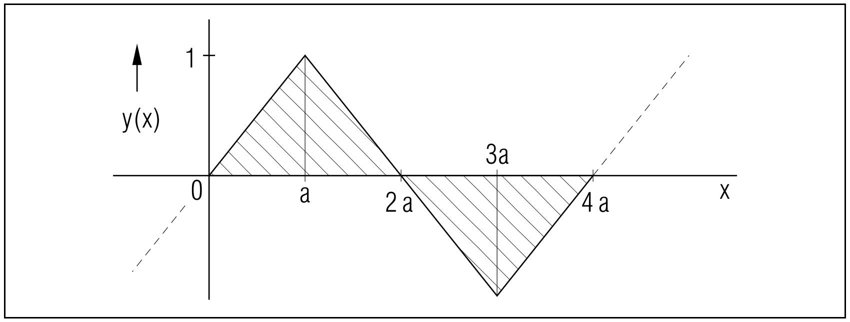

Figure 6. Graph of a saw-tooth function illustrating that the integral over the full cycle, 0 to $4a$, is zero. The area above the axis, from 0 to $2a$, is equal to the area below axis, from $2a$ to $4a$.

In the limit as $\Delta x \to 0$, the second area on the right goes to zero while the left side of the equation becomes dA/dx:

$$\frac{dA}{dx} = y(x).$$

Integrating this expression, the area under any $f(x)$ between any x_1 and x_2 is:

$$A(x_1, x_2) = \int_{x_1}^{x_2} y(x)\,dx.$$

For parts of the curve that are below the x-axis, the area between the curve and the x-axis is given a negative numerical value since the function is negative there. Thus for the graph of the "saw-tooth" function $y(x)$, shown in Fig. 6 in the interval from $x = 0$ to $x = 4a$, there is just as much area below the axis (negative area) as there is above the axis (positive area), so they cancel and the integral from 0 to $4a$ is zero. In this interval of x, the saw-tooth function has three straight lines: from $x = 0$ to $x = a$, from $x = a$ to $x = 3a$, and from $x = 3a$ to $x = 4a$. We integrate each of those lines and get the shaded area in Fig. 6:

$$\begin{aligned}\int_0^{4a} y(x)\,dx &= \int_0^a \frac{x}{a}\,dx + \int_a^{3a} \left(-\frac{x}{a} + 2\right)\,dx + \int_{3a}^{4a} \left(\frac{x}{a} - 4\right)\,dx \\ &= \frac{a}{2} + 0 - \frac{a}{2} = 0\,.\end{aligned} \tag{19}$$

▷ Show that each of the three linear functions in Eq. (19) does indeed properly describe its portion of the saw-tooth function (an easy way is

to plug in two different x values and show that each produces a y value that is obviously on the corresponding line). Then show that integration of each function gives the value shown.

There are many computer programs that can perform definite integrals numerically.[9]

Acknowledgments

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A. Logarithms

The logarithm of a number F to the base a is the power to which a must be raised to yield the number F:

$$\begin{aligned}\text{if:} &\quad F = a^x \\ \text{then:} &\quad \log_a F = x\ .\end{aligned}$$

For example, to the base $a = 10$, the logarithm of $F = 100$ is $x = \log_{10} F = 2$, while to the base $a = 2$, the logarithms of $F = 1,024$ and $F = 10$ are $x = \log_2 F = 10$ and 3.32, respectively. The familiar properties of logarithms, listed below, all follow from the definition.

If $F = a^x$ and $G = a^y$, then $FG = a^x a^y = a^{x+y}$, and

$$\log_a (FG) = x + y = \log_a F + \log_a G. \tag{20}$$

$$\log_a \frac{F}{G} = \log_a F - \log_a G \tag{21}$$

$$\log_a F^n = n \log_a F \tag{22}$$

$$\log_a a = 1 \tag{23}$$

$$\log_a 1 = 0 \tag{24}$$

Each of these can be proved using the definition of the logarithm. Another useful property expresses the relationship between the logarithms of the same number F in two different bases, a and b:

$$\log_b F = (\log_a F) \cdot (\log_b a). \tag{25}$$

[9] See "Numerical Integration" (MISN-0-349).

The inverse functional relationships,

$$x(F) = \log_a F; \qquad \text{and} \qquad F(x) = a^x$$

define a value of x for each given value of F, and vice versa.

B. Frequently Used Derivatives

$$\frac{d}{dx}(ax^n) = nax^{n-1} \tag{26}$$

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx} \tag{27}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df(x)}{dx} - f(x)\dfrac{dg(x)}{dx}}{[g(x)]^2} \tag{28}$$

$$\frac{d}{dt}[f(x)] = \frac{df}{dx}\cdot\frac{dx}{dt} \tag{29}$$

$$\frac{df(x)}{dx} = \left[\frac{dx(f)}{df}\right]^{-1} \tag{30}$$

Trigonometric

$$\frac{d}{d\theta}\sin\theta = \cos\theta \tag{31}$$

$$\frac{d}{d\theta}\sin m\theta = m\cos m\theta \tag{32}$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta \tag{33}$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta \tag{34}$$

$$\frac{d}{d\theta}\cot\theta = -\csc^2\theta \tag{35}$$

$$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \tag{36}$$

$$\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta \tag{37}$$

$$\frac{d}{d\theta}\sin^{-1}x(\theta) = \frac{\dfrac{dx}{d\theta}}{(1-x^2)^{1/2}} \tag{38}$$

$$\frac{d}{d\theta}\cos^{-1}x(\theta) = \frac{-\dfrac{dx}{d\theta}}{(1-x^2)^{1/2}} \tag{39}$$

$$\frac{d}{d\theta}\tan^{-1}x(\theta) = \frac{\dfrac{dx}{d\theta}}{1+x^2} \tag{40}$$

Logarithmic

$$\frac{d}{dx}\log_a U(x) = \left[\frac{1}{U(x)}\right][\log_a e]\left[\frac{dU(x)}{dx}\right] \tag{41}$$

$$\frac{d}{dx}\ell n\, x = \frac{1}{x} \tag{42}$$

$$\frac{d}{dx}(a^x) = a^x log_e a \tag{43}$$

$$\frac{d}{dx}\left(e^{y(x)}\right) = e^{y(x)}\left(\frac{dy(x)}{dx}\right) \tag{44}$$

PROBLEM SUPPLEMENT

Note: Problems 16-19 are used in this module's *Model Exam*.

1. Starting from the definition of the derivative, derive the expression for the derivative of $y = ax^n$, Text Eq. (26).
2. Again working directly from the definition, find the expression which gives the derivative of $f(x)/g(x)$ in terms of the derivatives of $f(x)$ and $g(x)$, Text Eq. (28).
3. Use the chain rule, Text Eq. (29), to determine the derivative of $F(x) = (ax^2 + bx + c)^{1/2}$.
4. Show that the derivative with respect to θ of $\tan\theta$ is $\sec^2\theta$, given the derivatives of the sine and cosine.
5. Given the derivative of $\ell n\, x$, Text Eq. (42), determine the expression for the derivative of e^x, simplified Text Eq. (44).
6. Evaluate the derivative of $(x^2 + 7x + 2)/(x + 2)$ at $x = 1$.
7. Evaluate the derivative of $e^{5x}(x^2 + 5x - 7)$ at $x = 0.2$.
8. Show that the derivative of $-\csc(4\theta - 1)$ is $4[\csc(4\theta - 1)][\cot(4\theta - 1)]$, given the derivatives of sine and cosine.
9. Show that the slope of the function $y(x) = 2x^3 - 3x^2 + 4$ at $x = -1$ is $+12$.
10. For the function in problem 9, show that the slope at $x = 1/3$ is $-4/3$.
11. For the function in problem 9, show that the locations of the maximum and minimum are, respectively, (0,4) and (1,3). With the aid of these results, sketch a graph of the function.
12. Referring to Appendix B, show that the integral

$$\int \left[Y(x)\frac{dZ(x)}{dx} + Z(x)\frac{dY(x)}{dx} \right] dx$$

is: $Y(x)Z(x) + C$.

13. Evaluate the integral of $6x^2 - 2$ and show that, if at $x = 2$ the integral is to have the value 10, then the integral's arbitrary constant is determined and the integral is $2(x^3 - x - 1)$.
14. Evaluate this definite integral: $\int_{-2}^{+2} (5x^2 + x)\, dx$.
15. A function $y(x)$ has these properties:

    (i) it is zero for all values of x up to $x = 5$, at which point it has the value $y = 10$.

    (ii) from $x = 5$ to $x = 20$ the function falls linearly (as a straight line) to zero, after which it is zero for all higher values of x.

    Sketch the function and evaluate its integral in the interval $x = 0$ to $x = 50$. (Hint: Make use of the geometrical interpretation of the integral.)
16. Verify that $f(t) = A\cos\omega t + B\sin\omega t$ (where A and B are constant and $\omega^2 = k/m$) is a solution of

$$m\frac{d^2 f(t)}{dt^2} + kf(t) = 0,$$

where $d^2 f(t)/dt^2$ is the second derivative of $f(t)$ and k and m are constants.

17. Evaluate the derivative of $(A\cos 5y)$ at $y = \pi/20$.
18. Evaluate this integral: $\int x^{1/2}\, dx$.
19. For each of the equations below, find the maximum and/or minimum points and distinguish between them.

    a. $y(x) = x^2 + x + 10$.

    b. $y(x) = x^3 - 3x + 2$.

    c. Determine A, B and C so the function $y(x) = Ax^3 + Bx^2 + C$ will have a minimum at $x = 1/3$.

Brief Answers:

3. $\dfrac{ax + b/2}{(ax^2 + bx + c)^{1/2}}$

6. 17/9
7. -66.33
14. 80/3
15. 75
17. $-5A/\sqrt{2}$
18. $(2/3)x^{3/2} + C$.
19. a. $\left(-\frac{1}{2}, \frac{39}{4}\right)$, minimum
    b. $(-1,4)$, maximum; $(1,0)$, minimum
    c. as long as $A = -2B$, the constants can be anything. *Help: [S-5]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-3d)*

Find the locations of the maximum and minima of the function:

$$y(x) = 5x^3 - 2x^2 - 3x + 2.$$

First, find dy/dx and set it equal to zero:

$$\frac{dy}{dx} = 15x^2 - 4x - 3 = 0$$

or

$$(5x - 3)(3x + 1) = 0.$$

Solving for the zeros, maxima or minima of the original function occur at: $x = 3/5; x = -1/3$. The corresponding values of y are found by inserting those x values into the original equation, e.g. at $x = 3/5$:

$$y = 5\left(\frac{3}{5}\right)^3 - 2\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right) + 2 = \frac{14}{25},$$

while at $x = -1/3, y = 70/27$. To find whether these points are maxima or minima, calculate the second derivative:

$$\frac{d^2y}{dx^2} = 30x - 4,$$

so at $x = 3/5$,

$$30\left(\frac{3}{5}\right) - 4 = 14 > 0.$$

Thus a minimum occurs at $(3/5, 14/25)$. At $x = -1/3$:

$$30\left(-\frac{1}{3}\right) - 4 = -14 < 0$$

so a maximum occurs at $(-1/3, 70/27)$.

S-2 *(from TX-2b)*

The binomial theorem states that

$$(c+d)^p = c^p + pc^{p-1}d + \frac{p(p-1)}{(1)(2)}c^{p-2}d^2 + \frac{p(p-1)(p-2)}{(1)(2)(3)}c^{p-3}d^3 + \ldots$$

so the expression $a(x+\Delta x)^p$ becomes $ax^p + apx^{p-1}\Delta x + \ldots + a\Delta x^p$. After subtracting ax^p and dividing by Δx, the only term that doesn't contain a factor of Δx is apx^{p-1}, so when the limit as $\Delta x \to 0$ is taken, this is the only non-zero term.

S-3 *(from TX-2f)*

Applying the trigonometric identity for $\sin A - \sin B$, where $A = kx + k\Delta x + \delta$ and $B = kx + \delta$, we obtain:

$$\sin(kx + k\Delta x + \delta) - \sin(kx+\delta) = 2\sin\left(\frac{k\Delta x}{2}\right)\cos\left(kx + \delta + \frac{k\Delta x}{2}\right).$$

After dividing this express by Δx and taking the limit as $\Delta x \to 0$, we get:

$$\frac{dy(x)}{dx} = \lim_{\Delta x\to 0} k\left(\frac{\sin\left(\frac{k\Delta x}{2}\right)}{k\frac{\Delta x}{2}}\right)\cos(kx + \delta + \frac{k\Delta x}{2}).$$

S-4 *(from TX-2e)*

The symbols used to express the chain rule in Sect. 2e are already used in other ways in this problem, so we make the substitutions $f \to F$, $x \to g$, and then $t \to x$ to get an equivalent chain rule:

$$\frac{dF(g(x))}{dx} = \left(\frac{dF}{dg}\right)\left(\frac{dg}{dx}\right).$$

Comparing the left side to the desired derivative, we make the correspondence:

$$g(x) = ax^2 + bx + c, \quad \text{so: } F(g) = g^{1/2}.$$

Then the chain rule gives:

$$\frac{dF}{dx} = \left(\frac{1}{2}g^{-1/2}\right)(2ax+b) = \frac{2ax+b}{2(ax^2+bx+c)^{1/2}}.$$

S-5 *(from PS-19c)*

The slope of the function has a zero at $x = 0$ and another at $x = -(2B)/(3A)$. The signs of the second derivatives show the first to be a maximum and the second a minimum. So just set $(1/3) = -(2B)/(3A)$.

MODEL EXAM

1. Verify that $f(t) = A\cos\omega t + B\sin\omega t$ (where A and B are constant and $\omega^2 = k/m$) is a solution of

$$m\frac{d^2 f(t)}{dt^2} + kf(t) = 0$$

where $d^2 f(t)/dt^2$ is the second derivative of $f(t)$ and k and m are constants.

2. Evaluate the derivative of $(A\cos 5y)$ at $y = \pi/20$.

3. Evaluate this integral: $\int x^{1/2}\,dx$.

4. For each of the equations below, find the maximum and/or minimum points and distinguish between them.

   a. $y(x) = x^2 + x + 10$.
   b. $y(x) = x^3 - 3x + 2$.
   c. Determine A, B and C so the function $y(x) = Ax^3 + Bx^2 + C$ will have a minimum at $x = 1/3$.

Brief Answers:

1. (A verification).

2. See this module's *Problem Supplement*, problem 17.

3. See this module's *Problem Supplement*, problem 18.

4. See this module's *Problem Supplement*, problem 19.