PHYSNET

MISN-0-508

ELECTROSTATIC ENERGY DENSITY

Electricity and Magnetism

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ELECTROSTATIC ENERGY DENSITY

by
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Title: **Electrostatic Energy Density**

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Input Skills:

1. Given dielectric media in conjunction with conducting surfaces, use the boundary value conditions to determine the potential, electric field and displacement in the media (MISN-0-507).
2. Employ double subscript matrix notation to represent sums involving two summation indices.

Output Skills (Knowledge):

K1. Vocabulary: electrostatic potential energy (of a charge distribution), electrostatic energy density.

K2. State the expression for the electrostatic energy in terms of the electric field and the electric displacement.

Output Skills (Problem Solving):

S1. Given a collection of external point charges or a distribution of surface and volume charge densities, calculate the electrostatic potential energy of the system of charges.

S2. Given a charge distribution in the presence of a dielectric medium, calculate the electrostatic energy density.

External Resources (Required):

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

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1. Introduction

This unit treats the electrostatic potential energy of an arbitrary charge distribution. This electrostatic energy is calculated as the work required to assemble this charge distribution from components of charge which are initially infinitely far away from each other. This definition will result in an expression for the electrostatic energy which involves an explicit integration over the charge distribution itself. The most important result of this unit consists in changing this expression for the electrostatic energy so that it involves a volume integral containing only the field vectors $\vec{E}$ and $\vec{D}$ of the system. This result will be of great importance in future units dealing with applications of Maxwell's equations.

2. Procedures

1. Read Chapter 6, Secs. 6-1 to 6-3, including the introductory paragraphs.

2. Write down the definition of electrostatic (potential) energy as given in the first sentence of Sec. 6-1.

3. Write down the expression for the electrostatic energy of an assembly of point charges (Eq. 6-6). Write down the electrostatic energy of a distribution of charge including free charge on conductors (Eq. 6-11). Alternatively, write down the electrostatic energy of a distribution of charge including free charge on conductors (Eq. 6-17) in terms of the field vectors $\vec{E}$ and $\vec{D}$. Write down the definition of (electrostatic) energy density as given in Eq. 6-18a and Eq. 6-18b. Include the conditions required for Eq. 6-18b to hold. You will be asked for one or more of these definitions and concepts on the exam covering this unit.

4. Read the Supplementary Notes for examples of the types of problems you must solve in Procedure 5.

5. Solve the following problems: 6-2, 6-11, 6-12.

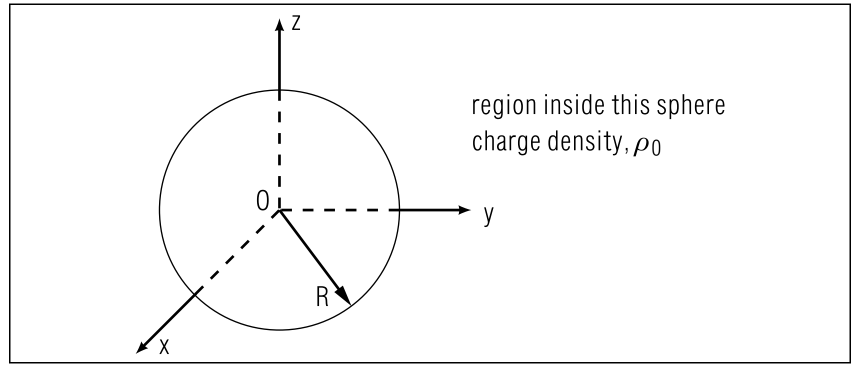

Figure 1. .

3. Supplementary Notes

3a. Problem 6-11. In order to solve this problem, the electrostatic potential and electric field are required. Let R be the radius of the sphere of uniform charge density ρ_0. Let the origin be at the center of the sphere (see Fig. 1). Apply Gauss' law to an imaginary spherical surface, concentric with the charged sphere and radius $r < R$. Thus,

$$\int \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int \rho\, dv\,,$$

$$E 4\pi r^2 = \frac{1}{\epsilon_0} \rho_0 \frac{4}{3} \pi r^3\,,$$

$$\Rightarrow E = \frac{\rho_0}{3\epsilon_0}\, r\,.$$

Thus,

$$V = -\frac{\rho_0}{6\epsilon_0} r^2 + V_0, \qquad r < R\,.$$

Perform the same calculation, but with $r > R$. Then,

$$\int \vec{E} \cdot \hat{n}\, da = \frac{1}{\epsilon_0} \int \rho\, dv\,,$$

$$E 4\pi r^2 = \frac{1}{\epsilon_0} \rho_0 \frac{4}{3} \pi R^3\,,$$

$$\Rightarrow E = \frac{\rho_0}{3\epsilon_0} \frac{R^3}{r^2}\,.$$

Thus,

$$V = \frac{\rho_0}{3\epsilon_0}\,\frac{R^3}{r^2} + V_0', \qquad r > R\,.$$

But, $V \to 0$ as $r \to \infty$ so that $V_0' = 0$. At $r = R$, the potential is continuous so,

$$-\frac{\rho_0}{6\epsilon_0}R^2 + V_0 = \frac{\rho}{3\epsilon_0}\,R^2\,,$$

so

$$V_0 = \frac{\rho_0}{2\epsilon_0}\,R^2\,.$$

Therefore, the complete expressions for the potential and field are:

$$V(r) = \begin{cases} -\dfrac{\rho_0}{6\epsilon_0}\,r^2 + \dfrac{\rho_0}{2\epsilon_0}R^2, & r \le R\,, \\ \dfrac{\rho_0}{3\epsilon_0}\,\dfrac{R^3}{r}, & r \ge R\,. \end{cases} \tag{1}$$

$$\vec{E}(\vec{r}) = E(\vec{r})\frac{\vec{r}}{r}\,,$$

where,

$$E(r) = \begin{cases} \dfrac{\rho_0}{3\epsilon_0}\,r, & r \le R\,, \\ \dfrac{\rho_0}{3\epsilon_0}\,\dfrac{R^3}{r^2}, & r > R\,. \end{cases}\;. \tag{2}$$

The solutions to part (a) and (b) of this problem now amount to integrating the expressions in Eqs. (6-11) and (6-18a), respectively using Eqs. (1) and (2) of these notes. So,

$$\begin{aligned} W &= \frac{1}{2}\int \rho\,V\,dv \\ &= \frac{1}{2}\int_0^R\int_0^\pi\int_0^{2\pi} \rho_0\left[-\frac{\rho_0}{6\epsilon_0}r^2 + \frac{\rho_0}{2\epsilon_0}R^2\right] r^2\sin\theta\,dr\,d\theta\,d\phi\,, \end{aligned}$$

and

$$\begin{aligned} W &= \frac{1}{2}\int \vec{E}\cdot\vec{D}\,dv = \int \frac{\epsilon_0}{2}E^2\,dv \\ &= \frac{\epsilon_0}{2}\int_0^R\int_0^\pi\int_0^{2\pi}\left[\frac{\rho_0}{3\epsilon_0}\,r\right]^2 r^2\sin\theta\,dr\,d\theta\,d\phi \\ &\quad + \frac{\epsilon_0}{2}\int_R^\infty\int_0^\pi\int_0^{2\pi}\left[\frac{\rho_0}{\epsilon_0}\,\frac{R^3}{r^2}\right]^2 r^2\sin\theta\,dr\,d\theta\,d\phi\,. \end{aligned}$$

Notice that one integral extends from 0 to R while the other goes from R to ∞. You job is simply to copy the above work and complete the integrations.

3b. Problem 6-12. Simply set $mc^2 = W$, where W is obtained in 6-11.

4c. Answers. 6-11: $R = 1.7 \times 10^{-15}$ m.

6-12: $\Delta W = \dfrac{q^2}{8\pi\epsilon_0}\left(\dfrac{K-1}{K}\right)\left(\dfrac{b-a}{ab}\right)$

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