PHYSNET

MISN-0-491

MATRICES AND TRANSFORMATIONS

Classical Mechanics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MATRICES AND TRANSFORMATIONS

by

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Title: **Matrices and Transformations**

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Input Skills:

1. Vocabulary: coordinate system, Cartesian coordinate system, reference frame, inertial frame orthogonal coordinate systems, rectangular coordinate system, dipolar cylindrical coordinate system, coordinate transformation, commutative.
2. State the geometrical definition of a vector (MISN-0-2).
3. Express the projection of a vector on a coordinate axis as a magnitude times an appropriate trigonometric function (MISN-0-2).

Output Skills (Knowledge):

K1. Vocabulary: matrix, unit matrix, diagonal matrix, transpose of a matrix, inverse of a matrix, orthogonal matrix, scalar, vector, tensor.

Output Skills (Rule Application):

R1. Carry out these operations involving matrices: multiplication, addition, subtraction, multiplication by a scalar.

R2. Determine the determinant and inverse of a square matrix.

R3. Determine the transpose and inverse of a product of matrices.

Output Skills (Problem Solving):

S1. Given a position coordinate or an arbitrary vector, perform a coordinate rotation both symbolically (in terms of matrices) and numerically.

External Resources (Required):

1. J. Marion, *Classical Dynamics*, Academic Press (1988).

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1. Introduction

In most introductory mechanics courses the emphasis is on the application of the laws of mechanics to various mechanical systems. It is tacitly, if not explicitly, assumed that the calculations are made entirely within one reference frame and, in fact, relative to a single coordinate system fixed in that frame. Usually no time is devoted to the problem of converting the description of the motion relative to one coordinate system into the appropriate description relative to another coordinate system. However, such conversion problems are important for at least two reasons. First, there is one of practicality - it is quite often advantageous, either for conceptual or calculational purposes, to move back and forth between two or more coordinate systems and/or reference frames. That is, the problem may be simpler in one system than another. Secondly, and more fundamentally, there is a very strong feeling among most practicing physicists that the laws of physics should have the same form in all inertial frames. This means that the laws of physics must be constructed in such a way that when they are transformed from one frame to another, their form does not change. Hence, it should be clear that a study of transformations is an important part of advanced mechanics. Certain kinds of transformations - those relating descriptions of vector quantities in rotated orthogonal coordinate systems - are very conveniently handled in matrix language while other more complicated transformations are best handled in the language of tensors. This unit is concerned with the fundamental ideas associated with matrices and transformations.

2. Procedures

1. Read Marion, Sections 1.1-1.9. Write down definitions of the quantities listed in Output Skill K1. Your definition of a matrix should include the multiplication rule, i.e.,

$$AB = C \Rightarrow \sum_j A_{ij} B_{jk} = C_{ik} \,.$$

In eq. (1.44) of Marion, a vector is defined to be a quantity which relative to an orthogonal coordinate system has three components and these three components behave in a certain way under coordinate rotations, namely:

$$A'_i = \sum_j \lambda_{ij} A_j \,.$$

Tensors are defined in a similar way. An n^{th} rank tensor is a quantity which has 3 components and transforms like a product of n vectors. (Note that vectors are first rank tensors while scalars are zeroth rank tensors). Thus under rotations a second rank tensor transforms like:

$$T'_{kl} = \sum_{ij} \lambda_{ki} \lambda_{lj} T_{ij} \,,$$

while a third-rank tensor transforms according to:

$$T'_{klm} = \sum_{hij} \lambda_{kh} \lambda_{li} \lambda_{mj} T_{hij} \,.$$

2. ▷ Work problem 1-4 in Marion.

   ▷ Work these exercises:

(1) Evaluate the product

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} .$$

(2) Given

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 5 & 2 \\ -2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 4 \\ -1 & -1 & 2 \\ 3 & 0 & 1 \end{pmatrix},$$

find:

$$A^2, AB, BA, B^2 \,.$$

(3) Using A and B of part (2), evaluate $(AB)^t$ and $B^t A^t$ separately and compare.

(4) Using A and B of part (2), evaluate det$A$, A^{-1}, det$B$, B^{-1}. Check your results by evaluating AA^{-1} and BB^{-1}.

(5) Using A and B of part (2) and the preceding results, evaluate:

$$\left(\frac{3}{5}A + BA^{-1}\right) B\,.$$

(6) Given:

$$P = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -1 & 6 & -3 \end{pmatrix}, \quad Q = \begin{pmatrix} -2 & 4 \\ 0 & 1 \\ 3 & -2 \\ -1 & 3 \end{pmatrix},$$

evaluate PQ and QP. If another column of numbers were added to Q so that it would have three columns and four rows, would PQ still be meaningful? How about QP?

3. ▷ Work problems 1-3 and 1-6 in Marion. If a certain point has coordinates $(x, y, z) = (2, 1, -3)$ in the original coordinate system of problem 1-1 of Marion, what are the coordinates of this same physical point in the rotated coordinated system?

The momentum of a particle as described in the original coordinate system of problem 1-3 of Marion is given by

$$\vec{p} = (2\hat{\ell}_1 - 5\hat{\ell}_2 + 3\hat{\ell}_3)\,\text{gm cm/sec}\,,$$

where the "hats" denote unit vectors. What is the momentum of this same particle as described in the rotated coordinate system? Are the physically important quantities, magnitude and direction, altered by the transformation?

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