PHYSNET

MISN-0-474

VACUUM FIELD EQUATIONS AND SCHWARZSCHILD'S SOLUTION

Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

VACUUM FIELD EQUATIONS AND SCHWARZSCHILD'S SOLUTION

by

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Title: **Vacuum Field Equations and Schwarzschild's Solution**

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Input Skills:

1. Unknown: assume (MISN-0-473).

Output Skills (Knowledge):

K1. Give a good plausibility argument for the form of the vacuum field equations of general relativity, including a definition of the Ricci tensor and a statement of its symmetry property.

K2. Establish the form of the Schwarzschild metric by: (a) writing down and justifying the general form of the metric exterior to a spherically symmetric static mass distribution subject to the constraints that (i) θ and ϕ are the usual spherical coordinate angles; (ii) r^2 is the square of the radial coordinate, equal to the proper area of a sphere concentric with the mass, divided by 4π; (iii) the metric is stationary, i.e. $\partial g_{\mu\nu}/\partial t$; (iv) the coordinates are orthogonal, i.e. cross terms do not occur in the metric; (b) Given the non-zero components of the Ricci tensor for the general spherically symmetric static metric to be able to determine the unknown functions so that the metric is a solution (the Schwarzschild solution) of the vacuum field equations.

K3. Concerning the Schwarzschild metric: (a) determine the geodesic equations in terms of r, θ, ϕ, t given the non-zero Christoffel symbols; (b) determine the difference between radar distance and ruler distance along a radial line; (c) explain the significance or lack of significance of the Schwarzschild radius; (d) show that the proper time required for a particle to fall from a finite height to the origin is finite while the coordinate time is infinite; (e) determine the coordinate transformation from the curvature coordinates (or Schwarzschild coordinates) to isotropic coordinates and interpret the isotropic coordinates.

External Resources (Required):

1. W. Rindler, *Essential Relativity*, van Nostrand (1977).

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1. Introduction

It was seen in MISN-0-472 that the main ingredients of General Relativity are 1) the equivalence principle, 2) the motion of free particles along geodesics and 3) the determination of the space-time structure by the distribution of mass. It would appear from a logical point of view that the last two cannot be independent since mass particles determine the space-time structure which in turn determines the motion of those same particles. This, in fact, is the case. It can be shown that the law of geodesic motion for free particles is contained within the field equations which determine the space-time structure from the mass distribution. Hence, the field equations are of paramount importance. This unit will thus begin the investigation of Einstein's field equations beginning with the simplest case - vacuum.

2. Procedures

1. Read section 8.2 of Rindler. The essential steps are:
   a. derivation of eq. 8.27.
   b. comparison of eq. 8.27 with 8.22 and identifying $R^{\mu}_{\nu\rho\sigma}U^{\nu}U^{\rho}$ with the second derivative of the Newtonian gravitational potential.
   c. recalling Laplace's equation for the potential in vacuum.
   d. generalizing Laplace's equation to $R^{\mu}_{\nu\rho\sigma}U^{\nu}U^{\rho} = 0$
   e. making the field equation independent of U^{ν}.
2. a. Read sections 23.2 and 23.3 of Misner, Thorne and Wheeler.
   Note: Compare eq. 8.36 of Rindler with eq. 23.7 of Misner, Thorne and Wheeler. The latter authors use a slightly different notation and a signature of $(-1, 1, 1, 1)$.
   b. Read section 8.3 of Rindler to the bottom of page 138.
   ▷ Exercise - Fill in any missing details between eqs. 8.37 - 8.41 and eq. 8.43 of Rindler.
   Read pages 138 - 141 of Rindler just for background information.

3. a. After much work (which you need not do) it is possible to show that the nonzero Christoffel symbols for the Schwarzschild metric are

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{\alpha^1}{2\alpha} \qquad (\alpha = 1 - 2\frac{m}{r})$$

$$\Gamma^1_{00} = \frac{1}{2}\alpha\alpha^1 \qquad \Gamma^1_{11} = -\frac{1}{2}\alpha^1/\alpha$$

$$\Gamma^1_{11} = -\frac{1}{2}\alpha^1/\alpha \qquad \Gamma^1_{22} = -r\alpha$$

$$\Gamma^1_{33} = -r\alpha\sin^2\theta \qquad \Gamma^2_{00} = -\sin\theta\cos\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

Note – In these expressions $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$.

▷ Exercise – Show that the geodesic equations for the Schwarzschild metric are (see MISN-0-473, p. 2)

$$\frac{d^2t}{d\lambda^2} + \frac{\alpha^1}{\alpha}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2}{d\lambda^2} + \frac{1}{2}\alpha\alpha^1\left(\frac{dt}{d\lambda}\right)^2 - \frac{\alpha^1}{2\alpha}\left(\frac{dr}{d\lambda}\right)^2 - r\alpha\left(\frac{d\theta}{d\lambda}\right)^2 - r\alpha\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0$$

▷ Exercise - Since $\alpha' dr/d\lambda = d\alpha/d\lambda$ show that the geodesic equation for t can be written in the form

$$\frac{d}{d\lambda}\left(\alpha\frac{dt}{d\lambda}\right) = 0$$

and hence has a solution of the form

$$\frac{dt}{d\lambda} = \frac{k}{\alpha}, \quad k = \text{ constant}$$

b. Read pages 141 - 142 of Rindler.

▷ Exercise - In arriving at equation 8.50 Rindler has assumed that light signals travel along radial lines in the Schwarzschild metric. This is equivalent to assuming that the relations

$$\frac{d\theta}{d\lambda} = \frac{d\phi}{d\lambda} = 0$$

and (from eq. 8.49)

$$\frac{dr}{d\lambda} = \alpha \frac{dt}{d\lambda}$$

satisfy the geodesic equations. Verify that this is the case.

▷ Exercise - Fill in any missing details in the analysis leading to eqs. 8.50 and 8.48 in Rindler.

c. - d. Read Section 8.8 of Rindler to the top of p. 153. Be sure you understand Lemaitre's coordinates (top of p. 151).

▷ Exercise - Show that eq. 8.83 of Rindler satisfies the geodesic equations for a massive particle. Hint: for a massive particle, the geodesic must be timelike ($dS^2 > 0$). Hence, the particle's proper time(s) may be used as a parameter in place of λ.

▷ Exercise – Fill in the details in the analysis leading to eq. 8.85 of Rindler. Rindler has assumed that the particle has unit rest mass so that $k = 1$.

(Optional) ▷ Exercise – Substitute eqs. 77.4 and 77.5 into eq. 77.3 of Rindler and recover the Schwarzschild metric.

e. ▷ Work Exercise 23.1a on p. 595 of Misner, Thorne and Wheeler. Use signature $(1, -1, -1, -1)$.

Acknowledgments

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