PHYSNET

MISN-0-470

COVARIANCE OF ELECTRODYNAMICS

Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

COVARIANCE OF ELECTRODYNAMICS

by

C. P. Frahm

Title: **Covariance of Electrodynamics**

Author: C. P. Frahm, Dept. of Physics, Illinois State Univ.

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Input Skills:

1. Define four-tensors in terms of their Lorentz transformation properties. Contract, add, and differentiate tensors. Show that the D'Alembertian is a Lorentz scalar (MISN-0-469).
2. State Maxwell's equations and define all the symbols involved.
3. Unknown (MISN-0-469).

Output Skills (Knowledge):

K1. Starting with Maxwell's equations: (a) define the scalar and vector potentials, and derive the wave equations for these potentials, (b) define the 4-vector potential, and justify the 4-vector character of each.

K2. (a) Define the electromagnetic field tensor and express it in terms of the electric and magnetic fields. (b) Express Maxwell's equations in a manifestly covariant form. (c) Derive and explain the significance of the continuity equation. (d) Express the Lorentz condition in a manifestly covariant form.

K3. Use the transformation character of the electromagnetic field tensor to determine the transformation rules for the electric and magnetic fields in special cases.

K4. Express the Lorentz force law in a manifestly covariant form and thereby obtain the transformation character of a 3-force.

External Resources (Required):

1. W. Rindler, *Essential Relativity*, Van Nostrand (1977).

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1. Introduction

According to the Relativity Postulate the laws of physics should take the same form in all inertial frames. Einstein was motivated to enunciate this postulate by consideration of electromagnetic phenomena - especially the interplay between electric and magnetic fields as a function of relative velocity. In fact, one might go so far as to say that relativity was "constructed" to make Maxwell's electromagnetic theory take on the same form in all inertial frames, thereby guaranteeing the constancy of the speed of light. Thus, electromagnetic theory provides an immediate ready-made example of the relativity postulate. For this reason this unit will be devoted to an investigation of those aspects of electrodynamics that bear directly on relativity. In particular the theory of Maxwell will be expressed in a tensor form which manifestly exhibits the invariance in form (or covariance, as it is called) of the theory. As very desirable by-products, this unit provides some practice in manipulation of tensors-and a transformation law for forces. The latter will prove very useful in the discussion of relativistic mechanics in the next unit.

Lorentz had earlier found that Maxwell's equations were invariant under the transformations which bear his name. However, he failed to grasp the significance of this result and he failed to take the step of replacing the Galilean transformation with the Lorentz transformation to connect inertial frames.

2. Procedures

0. (Review or introduction, whatever). The material of this unit requires all of Maxwell's theory of electromagnetic phenomena in vacuum. Hence a review of Maxwell's theory is called for. You may wish to consult your favorite E & M text although the sketch presented here should be sufficient for the needs of this course. Gaussian units (with $c = 1$) will be used throughout.

   a. To define the electric and magnetic fields, consider a small test charge q with velocity $\vec{u}$ (relative to our inertial frame) and located

at point $\vec{r}$ at time t. The total force experienced by the particle will consist in general of two parts - one which would be present even if the particle were uncharged ($q = 0$) and the other which is present only if the particle is charged ($q \neq 0$). The latter force is the electromagnetic force ($\vec{F}_{em}$) acting on the particle. The electromagnetic force in turn will generally consist of two parts - one which is present even if the particle is at rest ($\vec{u} = 0$) and one which is present only if the particle is moving ($\vec{u} \neq 0$). The former is called the electric force ($\vec{F}_e$) while the latter is called the magnetic force ($\vec{F}_m$). The electric and magnetic fields are then defined implicitly in terms of these two forces as follows:

$$\vec{F}_e(\vec{r}, t) = q\vec{E}(\vec{r}, t)$$

$$\vec{F}_m(\vec{r}, \vec{u}, t) = q\vec{u} \times \vec{B}(\vec{r}, t)$$

Taken together, they give the Lorentz force law

$$\vec{F}_{em} = q(\vec{E} + \vec{u} \times \vec{B})$$

   b. The electric and magnetic fields are generated by charges and currents. The relationships between the sources - charge density and current density J - and the electric and magnetic fields as well as the interrelation between these fields is given by Maxwell's equations (in vacuum):

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{J} + \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

It will not be necessary to have a deep physical understanding of these equations for this course. It will be sufficient to understand the mathematical symbols involved along with the definitions of $\vec{E}$, $\vec{B}$, ρ and $\vec{J}$.

1. a. For any vector field $\vec{B}$ having zero divergence (i.e. $\vec{\nabla} \cdot \vec{B} = 0$) there exists a vector field $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

In the case of the magnetic field (which has zero divergence by virtue of the fourth of Maxwell's equations) $\vec{A}$ is called the vector potential. It is not uniquely defined by the above expression since

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

where χ is any scalar function, would do just as well. A change from $\vec{A}$ to $\vec{A}'$ is called a gauge transformation. It does not alter the physics since

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$$

Substituting for $\vec{B}$ in the first of Maxwell's equations gives

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

or

$$\vec{\nabla} \times (\vec{E} + \partial \vec{A}/\partial t) = 0$$

Now any vector field that is irrotational (i.e. has zero curl) can be expressed as the gradient of a scalar function. Thus

$$\vec{E} + \partial \vec{A}/\partial t = -\vec{\nabla}\phi$$

or

$$\vec{E} = -(\vec{\nabla}\phi + \partial \vec{A}/\partial t)$$

The function ϕ is called the scalar potential and is implicitly defined by the previous equations.

▷ Exercise - Substitute for $\vec{E}$ and $\vec{B}$ in terms of the scalar and vector potentials into the remaining two of Maxwell's equations (the ones involving the sources) and show that

a)

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi J + \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t})$$

Hint: use the identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

b)

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t})$$

Note that both relations obtained in the exercise involve the quantity $\vec{\nabla} \cdot \vec{A} + \partial\phi/\partial t$. Because of the freedom allowed by gauge

transformations, it is always possible to choose $\vec{A}$ in such a way that this quantity is zero.

$$\vec{\nabla} \cdot \vec{A} + \partial\phi/\partial t = 0$$

This is called the Lorentz condition. Assuming the Lorentz condition has been satisfied, the two equations become

$$\nabla^2 \vec{A} - \partial^2 \vec{A}/\partial t^2 = -4\pi \vec{J}$$

$$\nabla^2 \phi - \partial^2 \phi/\partial t^2 = -4\pi\rho$$

These are both wave equations (propagation speed $c = 1$) with sources $\vec{J}$ and ρ.

▷ Exercise - In a region where $\vec{J} = 0$ show that

$$\vec{A} = \vec{a} e^{\imath(\vec{k}\cdot\vec{r} - \omega t)}$$

is a solution of the wave equation provided that

$$\frac{\omega^2}{k^2} = 1$$

The quantities $\vec{a}$, $\vec{k}$ and ω are constants.

Note: This is a plane wave of angular frequency ω propagating in the $\vec{k}$ direction.

b. By definition the charge density ρ is the charge per unit volume while the current density $\vec{J}$ is the charge crossing unit area in unit time in the direction of maximum flow. These two quantities together constitute a 4-vector-the current density 4-vector.

$$J^\mu = (\rho, \vec{J})$$

To verify the 4-vector character of J^μ it is convenient to consider a charged fluid flow having charge density ρ and 3-velocity $\vec{u}$. The current density is then given by

$$\vec{J} = \rho \vec{u}$$

If it is assumed that charge is a scalar quantity, then the charge density can be expressed in the form

$$\rho = \gamma_u \rho_0$$

where ρ_0 is the charge density in the rest frame of the fluid. The factor of γ_u arises from the Lorentz contraction of the volume in the direction of fluid flow. The quantity J^μ can then be written

$$J^\mu = (\gamma_u \rho_0, \gamma_u \rho_0 \vec{u}) = \rho_0 \gamma_u (1, \vec{u}) = \rho_0 U^\mu$$

where U^μ is the 4-velocity defined in MISN-0-469. Thus J^μ is a scalar (ρ_0) times a 4-vector (U^μ) and hence is itself a 4-vector.

The wave equations for the potentials $(\phi, \vec{A})$ can be expressed in the form:

$$\Box(\phi, \vec{A}) = -4\pi(\rho, \vec{J})$$

where the $\Box$ is the D'Alembertian operator defined in MISN-0-469. It was shown in MISN-0-469 that the D'Alembertian is a scalar operator and it has just been show in this unit that $(\rho, \vec{J})$ is a 4-vector. Thus since the electromagnetic theory of Maxwell is assumed to be a form invariant theory under Lorentz transformations, it can only be concluded that the quantity

$$A^\mu = (\phi, \vec{A})$$

is a 4-vector (the 4-vector potential). The wave equations can then be expressed in the compact and manifestly covariant form

$$\Box A^\mu = -4\pi J^\mu$$

Note: Do not confuse the 4-vector potential (A^μ) with the 4-acceleration (A^μ).

2. a. The electromagnetic field tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

It is a second rank tensor and it is antisymmetric in the sense that

$$F_{\nu\mu} = -F_{\mu\nu}$$

▷ Exercise - Show that

(a) $F_{01} = E_x$, $F_{02} = E_y$, $F_{03} = E_z$

(b) $F_{12} = -B_z$, $F_{23} = -B_x$, $F_{31} = -B_y$

Write down the matrix representation of $F_{\mu\nu}$ using its antisymmetry property and compare with the expression given on p. 7 of MISN-0-469.

b. ▷ Exercise - Show that the manifestly covariant form of Maxwell's equations is given by

a) $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$

b) $\partial_\rho F_{\mu\nu} + \partial F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$

Hints: In a) take $\nu = 0$ and obtain $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$. Then take $\nu = 1$ and obtain the x-component of

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{J} + \frac{\partial \vec{B}}{\partial t}$$

In b) take $\{\mu, \nu, \rho\} = \{1, 2, 3\}$ and obtain $\vec{\nabla} \cot \vec{B} = 0$. Then take $\{\mu, \nu, \rho\} = \{1, 2, 0\}$ and obtain the z-component of

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

c. Taking the 4-gradient of the source equations

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

gives

$$\partial_\nu \partial_\mu F^{\mu\nu} = 4\pi \partial_\nu J^\nu$$

Now since $F^{\nu\mu}$ is antisymmetric and $\partial_\nu \partial_\mu$ is symmetric, the left side is zero (think about this!). Thus

$$\partial_\nu J^\nu = 0$$

This is called the equation of continuity and is equivalent to a statement of conservation of charge.

▷ Exercise - a) Show that the equation of continuity can be expressed in the form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

b) Integrate this equation over a closed volume V and show that the time rate of change of the charge within the volume is equal to the rate at which charge flows out through the surface S bounding the volume V. (Use Gauss' law).

d. ▷ Exercise - Express the Lorentz condition in a manifestly covariant form.

3. By construction the electromagnetic field tensor is a second rank Lorentz tensor. Hence its behavior under Lorentz transformations is known.

$$F'_{\mu\nu}(x\rho') = F_{\lambda\omega}(x\rho)(a^{-1})^{\lambda}_{\mu}(a^{-1})^{\omega}_{\nu}$$

with

$$x\rho' = a\rho_{\eta} x^{\eta}$$

Note that the transformation law connects the field components in the two frames at the *same space-time point*. Also note that this transformation law can be conveniently expressed in matrix form

$$F' = (a^{-1})^T F a^{-1}$$

(Transformation laws for higher rank tensors, of course, cannot be expressed as matrix equations.)

▷ Exercise - Show that for a boost at speed v in the $+x$-direction, the electric and magnetic fields transform as follows:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - vB_z) \quad B'_y = \gamma(B_y + vE_z)$$

$$E'_z = \gamma(E_z - vB_y) \quad B'_z = \gamma(B_z + vE_y)$$

Note that the electric and magnetic fields become mixed in going from one inertial frame to another. Thus it is convenient to think of $\vec{E}$ and $\vec{B}$ as different manifestations of one and the same field - the electromagnetic field ($F_{\mu\nu}$).

▷ Exercise - Frame S' is boosted in the $+x$-direction at speed v relative to frame S. A charge q is at rest at the origin of frame S. What are the electric and magnetic fields in S' resulting from q at the instant the origins of the two frames coincide? Express in terms of the primed frame coordinates.

Read Rindler section 6.1, omit the long paragraph that begins on p. 97 Note that Rindler orders the components of the electromagnetic field tensor somewhat differently from this study guide. He also uses $A_{\mu\nu}$ instead of $F_{\mu\nu}$ for the field tensor.

Read Rindler, sections 6.2 - 6.4

4. Consider the following quantity

$$\mathcal{T}^{\mu} = g^{\mu\nu} F_{\nu\omega} U^{\omega}$$

By construction, it is a Lorentz 4-vector. As the following exercise shows, it is the manifestly covariant form of the Lorentz force law.

▷ Exercise - Show that the spatial part of $\mathcal{T}^{\mu}$ is γ_u times the Lorentz force per unit charge, i.e.

$$\vec{\mathcal{T}} = \gamma_u(\vec{E} + \vec{u} \times \vec{B}) = \gamma_u(\vec{F}_{em}/q)$$

This exercise yields valuable information on the transformation properties of 3 - forces. Clearly they must transforms like $1/\gamma_u$ times the spatial part of a 4-vector (since charge is a scalar).

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