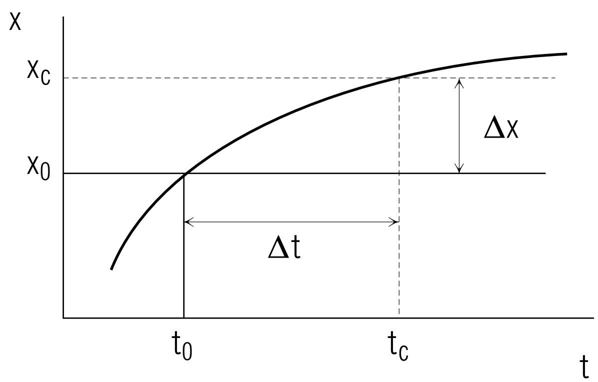
MISN-0-404

PHYSICAL CHANGES AND RATES



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PHYSICAL CHANGES AND RATES

by

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Title: **Physical Changes And Rates**

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Input Skills:

1. Vocabulary: variable, function, independent variable, dependent variable (MISN-0-403).

Output Skills (Knowledge):

K1. Define "change" and state two of its general properties.

K2. Define "rate," state two of its general properties and give the rates for $x = C$, $x = Ct$, $x = Ct^2$.

K3. Explain what is meant by "small enough" and how it is used to distinguish a difference from a derivative.

K4. Define "velocity," for motion in a straight line, in these cases: (a) v is constant; (b) v changes uniformly; (c) v is any function.

Output Skills (Problem Solving):

S1. Calculate the average rate of change of a variable, given its value at two points.

S2. Given the graph of a function, determine the rate of change of the function at any given point.

S3. Given a function of simple form, find the value or the expression representing its rate of change.

S4. Given the rate of change of a function which has a constant or uniformly changing rate, calculate total changes in the function.

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Abstract:
In trying to understand or predict the things which we observe, we commonly ask this question: If one thing is changed, how are other things affected? In this unit we discuss some useful ways for describing and calculating such corresponding changes in related quantities.*

> * Our discussion will examine, from a scientific point of view, some concepts which may already be familiar to you from the mathematics of the "calculus."

SECT.

A DESCRIPTION OF CHANGES

Consider two quantities t and x which are related so that x is a function depending on the independent variable t. Then any change in the quantity t is accompanied by a corresponding change in the quantity x, as illustrated in the graph of Fig. A-1. (For example, the relationship between t and x might specify how the distance x traveled by a car varies with the time t, or how the temperature x of the air varies with the height t above sea level.)

Suppose that we are interested in how x varies with t near some particular value t . When $t = t_0$, the function x has some corresponding value x_0 . For purposes of comparison, let us consider some changed value t_c of t where x has some corresponding value x_c. The "change" of the quantity t is then conventionally denoted by the symbol Δt and is defined as the difference:

$$\Delta t = t_c - t_0 \qquad \text{(A-1)}$$

(i.e., as the changed value minus the original value of t).*

> * The symbol Δt is to be regarded as a *single* algebraic symbol in which the capital Greek Letter Δ ("delta") is supposed to designate a difference.

Similarly, the corresponding change of the quantity x is denoted by Δx and defined as

$$\Delta x = x_c - x_0 \qquad \text{(A-2)}$$

Note that each of the changes Δt or Δx may be positive, negative, or zero (depending on whether the changed value is larger than, smaller than, or equal to the original value).

GENERAL PROPERTIES OF CHANGES

A quantity x may be functionally related to one or more other quantities, such as u or w. (For example, u might be $5t$ and w might be $2t^2$, or u and w might be unrelated to each other). How then does x change when the other quantities are changed?

Suppose that x is merely a constant multiple of some other quantity u so that

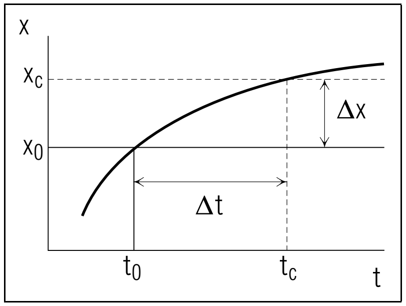


Fig. A-1: Comparison between the change Δt of a variable t and the corresponding change Δx of the function x.

$$x = Cu$$

where C is some constant (i.e., a quantity which remains unchanged in a given situation). What then is the change in x produced by some specified change in u? Consider some original value u where $x = x_0$ and some changed value u_c where $x = x_c$. Then the corresponding change in x is

$$\Delta x = x_c - x_0 = Cu_c - Cu_0 = C(u_c - u_0) = C\Delta u\,.$$

Thus,

$$\boxed{\text{if } x = Cu, \text{ then } \Delta x = C\Delta u\,.} \qquad \text{(A-3)}$$

This conclusion can be stated in words:

> The change in a constant multiple of a quantity is equal to the multiple of the corresponding change of this quantity. (A-4)

(For example, when prices change, the change in the cost of 5 apples is 5 times as large as the change in the cost of one apple.)

As another simple case, suppose that a quantity x is equal to the sum of two other quantities u and w so that

$$x = u + w\,.$$

What change in x is then produced by specified changes in the quantities u and w? Consider some original values u_0 and w_0 of these quantities, and some changed values u_c and w_c. Then the corresponding change in x is

$$\Delta x = x_c - x_0 = (u_c + w_c) - (u_0 + w_0) = (u_c - u_0) + (w_c - w_0)\,.$$

Thus,

$$\boxed{\text{if } x = u + w, \text{ then } \Delta x = \Delta u + \Delta w\,.} \qquad \text{(A-5)}$$

This conclusion can be stated in words:

> The change in a sum is equal to the sum of the individual changes. (A-6)

(For example, when prices change, the change in the total cost of a basket of apples and pears is equal to the change in the cost of the apples plus the change in the cost of the pears.)

ADVANTAGE OF CONSIDERING SMALL CHANGES

Suppose that a quantity x is somehow related to some other quantity t. Then it is usually easy to describe how x changes when t is changed by a small amount, although it may be quite difficult to find how x changes when t is changed by some large amount. For example, it may be quite simple to describe how the concentration x of one of the substances in a chemical reaction changes during any time interval Δt as small as 0.001 second. On the other hand, it might be a very complex task to find how the concentration x changes during a time interval as large as 3 hours.

An understanding of the relation between small changes can, however, be used as the basis of a very powerful method of prediction. For example, suppose that we can easily specify how a situation changes during any time interval Δt which is sufficiently small (say, 0.1 second). Then we can use information about the situation at some time t to predict the resulting situation 0.1 second later, i.e., at the time $t_0 + 0.1$ second. Then we can, in turn, use this information to predict the situation 0.1 second later, i.e., at the time $t_0 + 0.2$ second. Then we can, in turn, use this information to predict the situation 0.1 second later, i.e., at the time $t_0 + 0.3$ second. Proceeding in this way repetitively, we can thus continue making successive predictions until we arrive at predictions 1000 seconds after the time t_0, or 10 hours after t_0, or even a century after t_0! (With some ingenuity and the help of electronic computers, these successive calculations can often be carried out quite rapidly.)

In summary, simple information about small enough changes can be used repetitively to generate information about changes which are as large as one pleases. Hence a consideration of small changes is the basis of an extremely useful method of prediction which is used extensively in all the sciences. We shall illustrate this method more specifically in Sec. D and

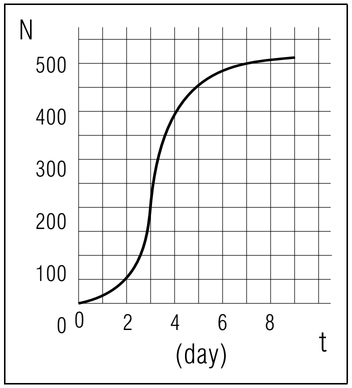


Fig. A-2.

shall use it frequently throughout the course.

Knowing About Related Changes

A-1 The graph in Fig. A-2 shows how the number N of *Paramecium aurelia* organisms in a fluid culture depends on the time t since the culture was begun. (a) Consider an original value $t_0 = 3$ day, and a change $\Delta t = 1$ day. What are the corresponding values of t_c, N_0, N_c, and ΔN? (b) Consider an original value $t = 5$ day, and the same change in time, $\Delta t = 1$ day. What are the corresponding values of t_c, N_0, N_c, and ΔN? *(Answer: 105)*

A-2 Five horizontal girders, each of length G, form a bridge section of total length $B = 5G$. After a cold night, the warm morning sun causes each girder to expand so that its length changes by $\Delta G = 0.50$ foot. What is the change ΔB in the total length of the bridge section? *(Answer: 112)*

SECT.

B DEFINITION OF RATE

Suppose that we wish to describe how some function x changes when the variable t changes near some particular value t_0. Then we can consider some changed value t_c of the variable so as to calculate the change $\Delta t = t_c - t_0$ and the corresponding change $\Delta x = x_c - x_0$. To compare these changes, we can then calculate their ratio:

$$\frac{\Delta x}{\Delta t} = \frac{x_c - x_0}{t_c - t_0} \qquad \text{(B-1)}$$

which is called the "*relative* change of x with respect to t."

Suppose that, starting with the same value t_0 of the variable, we considered a different changed value t'_c and thus a correspondingly different change $\Delta' t = t'_c - t_0$. (See Fig. B-1.) Then the corresponding change in x would be $\Delta' x = x'_c - x_0$. Ordinarily, the value of the relative change $\Delta' x/\Delta' t$ would then be different from that of $\Delta x/\Delta t$. In other words, the value of the relative change $\Delta x/\Delta t$ depends ordinarily on the size of the variable change Δt.

On the other hand, consider the special simple case where a graph of x versus t is a straight line near the value t_0 (as illustrated in Fig. B-2.) Suppose that we now again consider two different changes, a change $\Delta t = t_c - t_0$ and another change $\Delta' t = t'_c - t_0$. Since the triangles $P_0 P_c Q_c$ and $P_0 P'_c Q'_c$ in Fig. B-2 are similar, the proportionality of their corresponding sides then implies the simple result:

$$\frac{\Delta' x}{\Delta' t} = \frac{\Delta x}{\Delta t}$$

Thus we arrive at this conclusion:

> When a graph of x versus t is a straight line, the relative change $\Delta x/\Delta t$ has the same value independent of the value of the change Δt. (B-2)

Recall that every smooth curve (such as that in Fig. B-1) becomes indistinguishable from a straight line within a small enough region near any particular point. Hence the conclusion (B-2) implies that, for any function x which varies smoothly with t, the relative change $\Delta x/\Delta t$ has

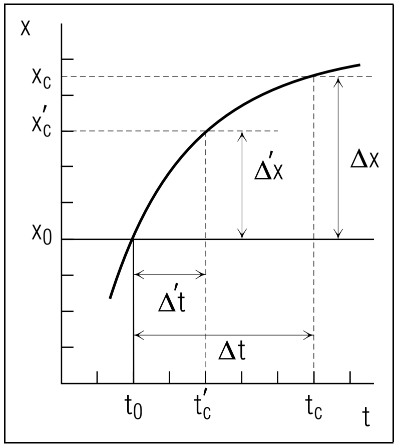


Fig. B-1: Dependence of the relative change $\Delta x/\Delta t$ on the value of Δt. (In this example, $\Delta x/\Delta t = 4.5/5.0 = 0.9$, while $\Delta' x/\Delta' t = 3.0/2.0 = 1.5$.)

a constant value provided that we restrict attention to changes Δt which are small enough.

The preceding conclusion illustrates that the relative change $\Delta x/\Delta t$ is particularly simple when it is considered for small enough changes Δt. Under these conditions the relative change is called a "rate" in accordance with this definition:

Def. | **Rate**: The "rate of change" (or "derivative") of x with respect to t, at the value t_0, is the ratio $\Delta x/\Delta t$ when the change $\Delta t = t_c - t_0$ is small enough (so that the value of $\Delta x/\Delta t$ remains constant for any smaller value of Δt). | (B-3)

It is convenient to indicate a difference which is "small enough" by the small letter d to distinguish it from a general difference (indicated by the Greek letter Δ) which can have any value. Thus one writes:

$$dt = \Delta t \text{ (if } \Delta t \text{ is small enough)} \qquad \text{(B-4)}$$

and correspondingly $dx = \Delta x$. With these abbreviations, the Def. (B-3) for the rate R of x with respect to t can be summarized by the equation:

$$\boxed{R = \frac{dx}{dt} \quad (\text{where } dt = t_c - t_0)} \qquad \text{(B-5)}$$

Note that the value of the rate R when $t = t_0$ involves a comparison of

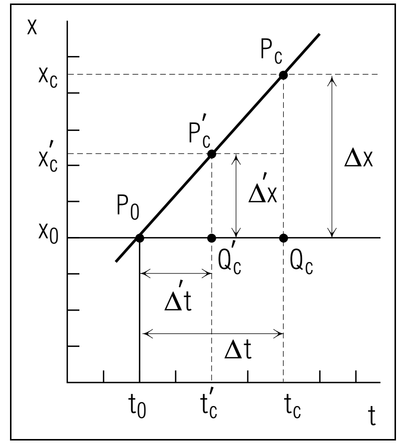


Fig. B-2: Dependence of the relative change $\Delta x/\Delta t$ on the value of Δt when the graph of x versus t is a straight line.

values of x in the vicinity of t . Hence this rate describes how x changes with t in the vicinity of t_0.

In the preceding discussion we have used the words "small enough" in accordance with this general criterion:

Def. | **Small enough**: The value of a quantity is said to be "small enough" (or "infinitesimal") if its replacement by *any* smaller value leaves certain specified results unchanged (within the desired precision). | (B-6)

In the case of the definition of rate, the specified result is the ratio $\Delta x/\Delta t$ and Δt is small enough if a graph of x versus t is straight within the interval Δt.

DISTINCTION BETWEEN SCIENTIFIC AND MATHEMATICAL DESCRIPTIONS

Scientists and mathematicians use different criteria for the concept "small enough." In a scientific description, any concept is only meaningful to the extent that it can be related to observations. Hence a quantity is considered small enough if any smaller value of this quantity leaves all results of interest unchanged within the desired or *observable* precision of description. On the other hand, mathematicians are only concerned with purely logical relationships irrespective of any observations. Hence they consider a quantity to be small enough if it is smaller than any *con-*

ceivable value, i.e., if it is essentially zero (although such a small value, say 10^{-20} meter, might never be observable). The existence of these different criteria for "small enough" implies that scientists use arguments based upon the mathematical criterion merely as approximations. Conversely, mathematicians regard arguments based upon the scientific criterion merely as approximations.

EXAMINATION OF THE DEFINITION OF RATE

Whenever one encounters an important new relation, such as the definition of rate, Def. (B-3), one must examine it with care to develop the understanding necessary to apply the relationship effectively in various contexts. Appendix B lists skills and knowledge which one needs in order to demonstrate a basic understanding of any relationship (although some of these items may not be relevant in certain cases). Let us then examine the definition of rate by trying to answer the pertinent questions about this definition.

► *Statement and example*

The definition of rate may be summarized compactly by the equation $R = dx/dt$ of Eq. (B-5). As a specific example, we may consider the motion of a small object (a "particle") along a straight line. The position of such a particle can then be specified by its "position coordinate" x, i.e., by its distance from some fixed point O on the line. The rate of change of the position coordinate with time t is called the "velocity" of the particle and is denoted by v. Thus the definition of velocity can be written as*

$$v = \frac{dx}{dt} \qquad \text{(B-7)}$$

* A more general definition of velocity will be given in a later unit.

For example, Fig. B-3 illustrates the motion of a particle falling vertically downward near the surface of the earth. Then the velocity v of the particle at the time $t = 2.0$ second can be found approximately from the graph in Fig. B-3b by calculating the ratio dx/dt with the indicated values of dx and dt (which are small enough so that the graph is nearly a straight line in the small interval dt). Thus we find:

$$v = \frac{dx}{dt} \approx \frac{4\,\text{meter}}{0.2\,\text{second}} \approx 20\,\text{meter/second}$$

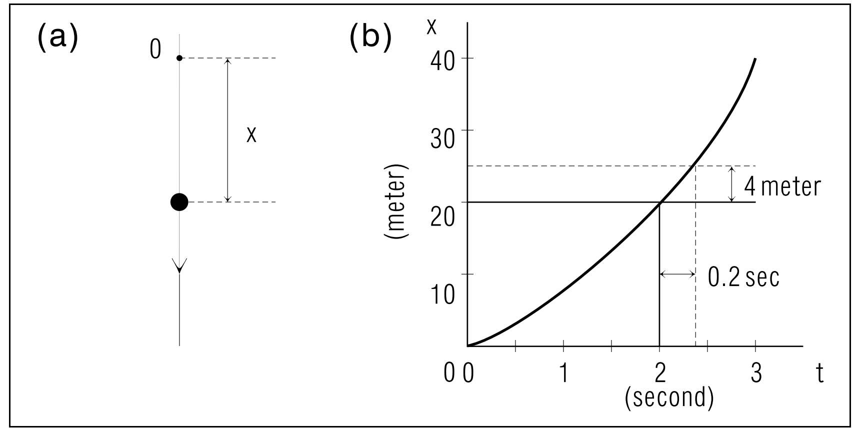


Fig. B-3: Motion of a falling particle. (a) Vertical path and position coordinate x of the particle. (b) Graph of x versus the time t.

Table B-1 illustrates a sample calculation of v with more precise numerical data. For the specified value $t_0 = 2.00$ second, this calculation is repeated with successively smaller values of Δt. When Δt is made small enough, the ratio $\Delta x/\Delta t$ no longer changes appreciably. Thus, a value of $\Delta t = 0.01$ second is small enough to infer that $v = dx/dt = 20.0$ meter/second.

Δt (sec)	Δx (meter)	$\Delta x/\Delta t$ (meter/sec)
0.200	4.200	21.0
0.100	2.050	20.5
0.0600	1.218	20.3
0.0200	0.402	20.1
0.0100	0.200	20.0

Table B-1: Sample calculation of the ratio $\Delta x/\Delta t$ for successively smaller values of $\Delta t = t_c - t_0$ for the fixed value $t_0 = 2.00$ second.

► *Properties*

The definition dx/dt of the rate implies that the rate is a number which can be positive, negative, or zero since each of the differences dx or dt can be positive, negative, or zero.*

* If $dt = 0$ and $dx \neq 0$, the rate is said to be infinite.

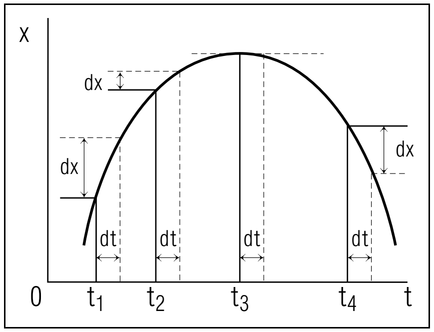


Fig. B-4: Rate dx/dt calculated for several values of t. The rate is positive at t_1 and t_2, zero and t_3, and negative at t_4. (The magnitude of dt has been exaggerated to make it easily visible.)

The unit of the rate dx/dt is equal to the unit of x divided by the unit of t (e.g., meter/second if x denotes a distance and t a time).

► *Interpretation using graphs*

The graphical interpretation of the rate becomes apparent by looking at a graph, such as that in Fig. B-4, and considering the rate calculated at various values of t (such as t_1, t_2, t_3, and t_4). We may compare each such value with a neighboring value slightly larger by the same positive difference dt. In each case we may then use the graph to determine the corresponding difference dx and thus to calculate the rate dx/dt. A look at Fig. B-4 then leads to these conclusions: The magnitude of dx, and hence that of the rate dx/dt, is large for values of t (such as t_1 or t_4) where the graph is rising or falling steeply; it is less large for values of t (such as t_2) where the graph is rising or falling less steeply; and it is equal to zero for values of t (such as t_3) where the graph is locally horizontal (so the change dx is indistinguishable from zero for a small enough change dt).

Furthermore, the *sign* of dx, and hence that of the rate dx/dt, is positive for values of t (such as t_1) where x increases with increasing t (so that the graph goes upward); and it is negative for values of t (such as t_4) where x decreases with increasing t (so that the graph goes downward).

In summary, the *magnitude of the rate* dx/dt at t_0 describes how rapidly x changes with t near t_0 (i.e., the steepness of the graph of x versus t). The *sign of the rate* at t_0 (positive, negative, or zero) describes whether x increases, decreases, or remains constant with increasing t near t_0 (i.e., whether the graph goes upward, downward, or remains horizontal). The analogy between a graph and the shape of a hill suggests why the rate

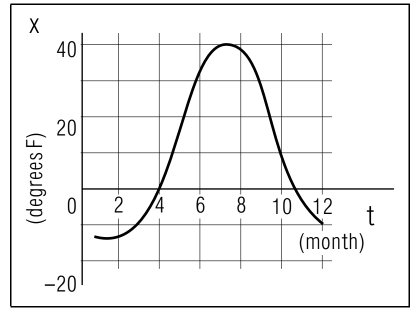


Fig. B-5: Variation of the average temperature x with time t in Barrow, Alaska. The temperature is expressed in degrees Fahrenheit. The time $t = 1$ month corresponds to January, $t = 2$ month to February, and so on.

dx/dt at a particular value t_0 is commonly called the "slope of the graph" at the point corresponding to t_0.

► *Meaning of symbols*

The meaning of the symbols appearing in the definition dx/dt of the rate is specified by some important conditions Thus the changes dt and dx must be small enough. Furthermore, dt and dx must be differences calculated at the particular value t at which the rate is to be found.

► *Comparing rate and function*

For any value of t, it is important to distinguish carefully between the value of the rate dx/dt and the value of the function x. Thus the rate dx/dt is related directly to the corresponding changes of the variables, rather than to the values of the variables themselves. Hence any value of x may be associated with any value of the rate dx/dt. For example, when $x = 0$, the rate dx/dt may be positive or negative (e.g., for the values $t = 4$ month and $t = 11$ month in Fig. B-5). Similarly, when x is positive, the rate dx/dt may be positive, zero, or negative (e.g., for the values $t = 5$ month, $t = 7.5$ month, and $t = 10$ month in Fig. B-5.)

► *Relating quantities*

The definition dx/dt of the rate allows us to find the value of any quantity in this relationship from the others. Thus it implies that $dx = Rdt$, or that $(x_c - x_0) = R(t_c - t_0)$ if the differences $dx = x_c - x_0$ and $dt = t_c - t_0$ are small enough.

(For example, if x denotes the position coordinate and t the time, the definition $v = dx/dt$ of the velocity implies that $dx = vdt$, i.e., that the small change dx in the position coordinate is merely equal to the velocity v multiplied by the small elapsed time interval dt.)

► *Dependence*

According to the definition of rate, the changes dx and dt are supposed to be small enough so that the ratio dx/dt has a value independent of the size of dx and dt. Hence dx is proportional to dt as long as these changes remain small enough. (For example, if dt were 3 times as large, the corresponding change dx would also be 3 times as large.)

The preceding examination of the definition of rate should enable you to develop an understanding of this relation. To demonstrate such an understanding, you should be able to answer questions like this:

Understanding the Definition of Rate (Cap. 1)

B-1 *Statement and example:* A dog runs up a hill and so his horizontal distance from a fire hydrant changes by the small enough amount $dx = 0.1$ meter. The corresponding change in his height y above the fire hydrant is $dy = 0.05$ meter. (a) State the definition of the rate R of change of y with respect to x. (b) What is the value of R for the situation described? *(Answer: 101)*

B-2 *Properties:* A function T depends on a variable h. (a) Use these symbols to write a symbol for the rate of change of T with respect to h. (b) What are the possible signs (positive, negative, zero) of this rate? (c) If T is a temperature with the unit (degree Centigrade) and h is a height with the unit meter, what is the unit of the rate of T with respect to h? *(Answer: 115)*

B-3 *Interpretation:* Nuclei of technetium-99 (a radioactive substance commonly used in biological investigations) "decay" by emitting "positron" particles which can be detected. Thus the number N of technetium nuclei in a sample decreases with time t as shown in Fig. B-6. If the time interval 5.00 minute is small enough, what is the rate dN/dt when the time is 10 minute? *(Answer: 109) ([s-1], [p-1], [p-2])*

B-4 *Meaning of quantities:* For each of the following examples, either find the value of the indicated rate, or briefly describe why the information provided can not be used to find this rate. (a) A pilot announces to the passengers of an airplane that they are flying at a height $h = 30{,}000$ foot, and that the temperature outside is $T_0 = -12$ (degree Fahrenheit). What is the rate dh/dT at T_0? (b) In Fig. B-7, the graph and table indicate how the speed v of blood in a human femoral artery depends on the time t. What is the rate dv/dt at the time $t = 0.1$ sec (with

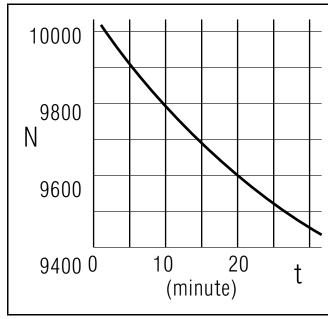


Fig. B-6.

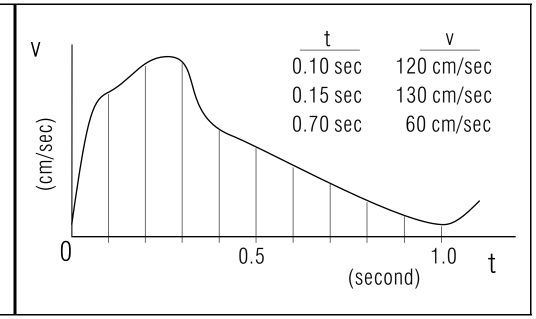


Fig. B-7.

the best precision possible using the information provided)? *(Answer: 120) (Suggestion: [s-7])*

B-5 *Comparison of functions, changes, and rates:* (a) In the graph shown in Fig. B-3b, what is the value of the *function* x corresponding to the time $t = 2$ sec? (b) What is the value of the *change in the function* x corresponding to the change $dt = 0.2$ sec shown in Fig. B-3b? (c) Using the small enough value of dt shown, what is the value of the rate of change of x with respect to t for $t = 2$ sec? (d) For which of the values of t shown in Fig. B-4 is x positive while the corresponding value of dx/dt is negative? *(Answer: 118)*

B-6 *Relating quantities:* In 1970 the population of the United States was 2.03×10^8, and the rate of change of population with time was 2.4×10^6 year^{-1}. (a) If a time interval of 5.0 year is small enough, use this information to find the change in the US population between 1970 and 1975. (b) What is the resulting population in 1975? *(Answer: 119) ([s-9], [p-3])*

Using Graphs to Describe Rates (Cap. 2)

B-7 Use the graph shown in Fig. B-5 to determine the sign $(+, -, 0)$ of the function x and of the rate dx/dt corresponding to each of these values of t: 2 month, 4 month, 10 month. *(Answer: 106) ([s-8], [p-4])*

B-8 The graph in Fig. B-8 shows the relation between air temperature T and horizontal distance x from a vertical burning surface. In each of the following pairs of values for x, for which value is the corre-

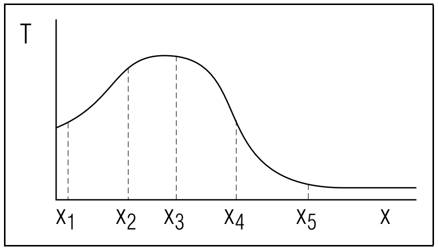


Fig. B-8.

sponding value of the rate dT/dx larger? (a) x_1, x_2 (b) x_2, x_3 (c) x_3, x_4 (d) x_4, x_5 (e) x_1, x_4. *(Answer: 116) ([s-11], [p-5])*

SECT.

C FINDING RATES FROM FUNCTIONS

To calculate the rate dx/dt for a particular value $t = t_0$, we merely need information about how x depends on t in the immediate vicinity of t_0. When the information about the relationship between x and t is available in numerical or graphical form, the rate dx/dt may be calculated in the manner already illustrated in connection with Fig. B-3 or Table B-1. But when the relationship between x and t is expressed as a formula, it is possible to derive from it a formula for the rate. In this case it is particularly easy to find values of the rate or to examine its properties. Let us then discuss a few particularly simple and useful cases where formulas for the rate can readily be found.

(1) As the simplest example, suppose that $x = C$, where C is some constant independent of t. Then the value of x remains unchanged, irrespective of any change in t. Therefore $dx = 0$ for any small change dt, so that $dx/dt = 0$. Thus

$$\boxed{\text{if } x = C,\ \frac{dx}{dt} = 0} \qquad \text{(C-1)}$$

i.e., the rate of change of a constant is zero.

(2) As another example, suppose that x is a constant multiple of t so that

$$x = Ct \qquad \text{(C-2)}$$

where C is some constant. Then we know from Eq. (A-3) or statement (A-4) that any change in x is just C times as large as the corresponding change in t. Thus $\Delta x = C\Delta t$ or, for changes which are small enough, $dx = Cdt$. Hence,

$$\boxed{\text{if } x = Ct,\ \frac{dx}{dt} = C} \qquad \text{(C-3)}$$

In other words, the rate dx/dt has then a constant value equal to C for all values of t. Fig. C-1a shows graphically how the function $x = Ct$ depends on t and how the corresponding rate dx/dt depends on t.

(3) As a last example, suppose that x depends on t so that

$$x = Ct^2 \qquad \text{(C-4)}$$

where C is some constant. To calculate the rate dx/dt for a particular

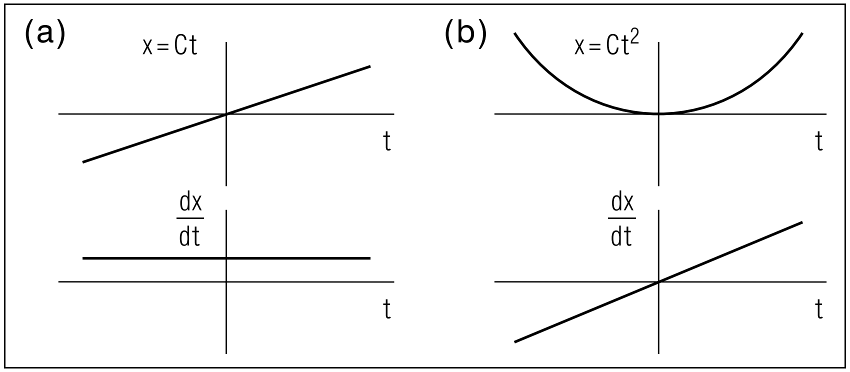


Fig. C-1: Graphs showing the dependence of x on t and the corresponding dependence of the rate dx/dt on t. (a) $x = Ct$. (b) $x = Ct^2$.

value t_0, we start by finding the change $\Delta x = x_c - x_0$ produced by any change $\Delta t = t_c - t_0$. By Eq. (C-4) we obtain

$$\Delta x = x_c - x_0 = Ct_c^2 - Ct_0^2 = C[t_c^2 - t_0^2] \qquad \text{(C-5)}$$

To express Δx in terms of $\Delta t = t_c - t_0$, we note that $t_c = t_0 + \Delta t$ so that Eq. (C-5) becomes

$$\Delta x = C[(t_0 + \Delta t)^2 - t_0^2] = C[t_0^2 + 2t_0\Delta t + (\Delta t)^2 - t_0^2]$$

or

$$\Delta x = C[2t_0\Delta t + (\Delta t)^2] \qquad \text{(C-6)}$$

since the terms t_0^2 and $-t_0^2$ cancel each other. Hence

$$\frac{\Delta x}{\Delta t} = C[2t_0 + \Delta t] \qquad \text{(C-7)}$$

To find the rate, we must use a value of Δt small enough so that the ratio $\Delta x/\Delta t$ does not depend on Δt. Thus we must use a value of Δt small enough so that the right side of Eq. (C-7) becomes indistinguishable from $C[2t_0]$. If the small enough changes are denoted by dx and dt, the result Eq. (C-7) becomes then

$$\frac{dx}{dt} = 2Ct_0 \qquad \text{(C-8)}$$

The preceding argument holds for *any* value of t_0. Hence we can conclude quite generally that, for any particular value t of the variable,

$$\boxed{\text{if } x = Ct^2,\ \frac{dx}{dt} = 2Ct} \qquad \text{(C-9)}$$

Figure C-1b shows graphically how the function $x = Ct^2$ depends on t and how the corresponding rate dx/dt depends on t. Note that the slope of the graph of x versus t is negative when t is negative, is zero when $t = 0$, and is positive when t is positive. Furthermore, the *magnitude* of the slope (i.e., the steepness of the graph) increases as the *magnitude* of t increases. The formula Eq. (C-9) for the rate, and its corresponding graph in Fig. C-1b, show that the rate dx/dt varies with t in a manner consistent with these conclusions.*

* If $x = Ct^n$, $dx/dt = nCt^{n-1}$ for all values of n, as verified by Rule (C-1), Rule (C-3), and Rule (C-9) for $n = 0$, 1, or 2.

APPLICATION TO MOTION

Let us apply our results to the motion of a particle along a line so that x denotes its position coordinate and t the time. Suppose that $x = C$ so that the position coordinate remains unchanged, i.e., so that the particle is at rest. Then Rule (C-1) implies that the velocity $v = dx/dt = 0$. Suppose that $x = Ct$ so that the position coordinate changes uniformly with time. Then Rule (C-3) implies that $v = dx/dt = C$, i.e., that the particle moves with *constant* velocity. Finally, suppose that $x = Ct^2$ where C is some positive constant. Then Rule (C-9) implies that $v = dx/dt = 2Ct$ so that the velocity of the particle *increases* uniformly with time.

The general properties discussed in Sec. A are, of course, also true for changes which are small enough. For example, if Rule (A-3) is applied to small enough changes and then divided by the corresponding small change dt, we obtain the result

$$\boxed{\text{if } x = Cu,\ \frac{dx}{dt} = C\frac{du}{dt}} \qquad \text{(C-10)}$$

In other words, the rate of a constant multiple of a function is simply equal to the multiple of the rate of this function. Similarly, Rule (A-5) implies that

$$\boxed{\text{if } x = u + w,\ \frac{dx}{dt} = \frac{du}{dt} + \frac{dw}{dt}} \qquad \text{(C-11)}$$

In other words, the rate of a sum is simply equal to the sum of the individual rates.

Example C-1: Growth rate of a child

Statistical data provide information about how the average height h of a child increases with time t (measured from birth). For an American male child between the ages of 2 and 11 years, these data can be summarized by this equation:

$$h = (69\,\text{cm}) + (9.7\,\text{cm/yr})t - (0.26\,\text{cm/yr}^2)t^2 \qquad \text{(C-12)}$$

where cm = centimeter and yr = year. The growth rate $R = dh/dt$ of a child at any time t can then be found by adding the rates for each of the terms on the right side of Eq. (C-12). By using Rule (C-1), Rule (C-3), and Rule (C-9) we find

$$R = 0 + (9.7\,\text{cm/yr}) - 2(0.26\,\text{cm/yr}^2)t$$

or

$$R = (9.7\,\text{cm/yr}) - (0.52\,\text{cm/yr}^2)t \qquad \text{(C-13)}$$

This formula indicates (because of the minus sign) that the growth rate decreases with increasing age t. For example, at age $t = 2.0$ year, when Eq. (C-12) implies that $h = 87$ cm, Eq. (C-13) predicts that the growth rate is

$$R = (9.7\,\text{cm/yr}) - (0.52\,\text{cm/yr}^2)(2.0\,\text{yr}) = 8.7\,\text{cm/yr} \qquad \text{(C-14)}$$

But at age 10 years, when $h = 140$ cm, Eq. (C-13) predicts that R is only 4.5 cm/yr.

The above discussion of finding rates using simple formulas should enable you to acquire this capability:

Finding Rates From Simple Formulas (Cap. 3)

C-1 What is the rate of change with respect to t of each of the following functions (x, y, and s)? Express your answers using t and constant quantities (including numbers with units). (a) $x = (5\,\text{meter/sec}^2)t^2$, (b) $y = 3$ sec, (c) $s = (1/2)a_0t^2 + v_0t + x_0$, where a_0, v_0, and x_0 are constant quantities. *(Answer: 102) ([s-10], [p-6])*

C-2 When combined with the enzyme urease, urea and water react to form ammonia and carbon dioxide. During the first minutes after combination, the following approximate formula relates the number

N of ammonia atoms in a mixture to the time t since the reactants were combined:

$$N = (3.0 \times 10^{18}\,\text{minute}^{-1})t - (6.0 \times 10^{16}\,\text{minute}^{-2})t^2$$

(a) Write an expression for the rate dN/dt in terms of the variable t and constant quantities. (b) What is the value of dN/dt for $t = 0$ minute and for $t = 10$ minute? (c) After several hours, the preceding formula no longer applies, and instead N remains constant. What then is the value of dN/dt? *(Answer: 111) ([s-3], [p-7])*

SECT.

D FINDING FUNCTIONS FROM RATES

In the preceding section we discussed how one can use information about a function to find information about its rate. Let us now examine the frequently occurring case where one wants to do the opposite, i.e., to use available information about a rate to find information about the function.

Information about the rate $R = dx/dt$ allows us to find the small change $dx = Rdt$ corresponding to any small enough change dt in the variable t. Suppose then that we have information about the rate R for all values of t in the interval between some initial value t_A and some final value t_B. Then we can subdivide this interval into some large number n of successive small enough intervals d_1t, d_2t, $\ldots$, d_nt. (For simplicity, we may choose all these small intervals to be of equal size.) For each small change dt, we may then calculate the corresponding change $dx = Rdt$. (For example, knowing the rate R_2 when $t = t_2$, we can calculate the change $d_2x = R_2d_2t$ in x when t changes by the small amount $d_2t = t_3 - t_2$.) The *total* change $\Delta x = x_B - x_A$ of the function x, corresponding to the *total* change $\Delta t = t_B - t_A$ of the variable t, is then simply the sum of all the successive small changes of the function. Thus:*

$$\Delta x = d_1x + d_2x + \ldots + d_nx \qquad \text{(D-1)}$$

or

$$\Delta x = R_1d_1t + R_2d_2t + \ldots + R_nd_nt \qquad \text{(D-2)}$$

* The changes dt used in this calculation are small enough if the use of smaller values of dt would not affect the total change Δx (within the desired precision).

If the initial value x_A of the function is known, the total change $\Delta x = x_B - x_A$ calculated by Eq. (D-2) can then immediately be used to find the final value x_B of the function. Thus $x_B = x_A + \Delta x$.

In some simple cases, the sum in Eq. (D-2) can be readily calculated. To illustrate the general procedure, we shall consider as a specific example the motion of a particle along a straight line so that x denotes the position coordinate of the particle at any time t. Then we should like

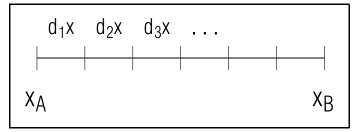


Fig. D-1: Successive small displacements of a particle moving with constant velocity.

to use information about the rate $v = dx/dt$ (i.e., about the velocity) to deduce information about the position coordinate x. (The change Δx in the position coordinate is called the “displacement” of the particle. The magnitude of this displacement is the net distance traveled by the particle.)

MOTION WITH CONSTANT VELOCITY

Suppose that the velocity of the particle has a *constant* value v at all times between t_A and t_B. Then the definition $v = dx/dt$ of the velocity implies that *all* corresponding small changes are related by $dx = vdt$ with the *same* value of v. Hence the corresponding *total* changes $\Delta x = x_B - x_A$ and $\Delta t = t_B - t_A$ should be similarly related by:*

$$\boxed{\Delta x = v\Delta t} \qquad \text{(D-3)}$$

* Indeed, the constancy of the rate v implies that a graph of x versus t is a straight line. Hence *any* change is small enough so that $\Delta x/\Delta t = dx/dt = v$.

To verify this result explicitly, note that the total change Δx in the position coordinate of the particle is merely the sum of all the successive small changes indicated in Fig. D-1. Thus:

$$\Delta x = d_1x + d_2x + \ldots$$

or

$$\Delta x = vd_1t + vd_2t + \ldots = v(d_1t + d_2t + \ldots) = v\Delta t$$

since the value of v is always the same and since the sum of all the successive small changes in the time is just the total change Δt in the time.

The conclusion expressed mathematically in Eq. (D-3) can be expressed in words:

If the velocity of a particle is *constant*, the total change in the position coordinate of the particle is simply equal to the velocity multiplied by the total change in time. (D-4)

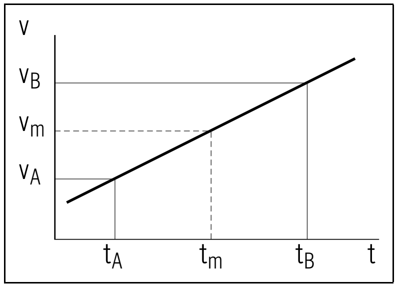


Fig. D-2: Graph showing the dependence of a uniformly changing velocity v on the time t.

Example D-1: Car moving with constant velocity

Suppose that a car moves along a straight road with a *constant* velocity $v = 45\,\text{mile/hour}$ during a time interval $\Delta t = 2.0\,\text{hour}$. Then the displacement Δx of the car during this time is simply $v\Delta t = (45\,\text{mile/hour})(2.0\,\text{hour}) = 90\,\text{mile}$.

MOTION WITH UNIFORMLY CHANGING VELOCITY

The velocity v of a particle is said to change "uniformly" with time if it changes at a constant rate (i.e., if dv/dt is constant). Then the velocity v changes by equal amounts in equal amounts of time and a graph of v versus the time t must be a straight line (such as that indicated in Fig. D-2). What then is the total displacement Δx of the particle during some time interval $\Delta t = t_B - t_A$?

Imagine that the time interval between t_A and t_B is subdivided into many very small time intervals of equal size dt. If v increases uniformly with time (as illustrated in Fig. D-2), the successive small displacements $dx = vdt$ of the particle then also increase uniformly as illustrated in Fig. D-3a and Fig. D-3b. But this last figure indicates that the amount by which d_5x is *larger* than the middle displacement d_4x is equal to the amount by which d_3x is *smaller* than d_4x.

Any other pair of corresponding displacements (indicated by the arrows in Fig. D-3b) differ from each other by similar compensating amounts when compared with the middle displacement d_4x. The net result is that the sum of all the successive displacements is the same as if all the displacements had (as shown in Fig. D-3c) the same constant value equal to that of the middle displacement $d_4x = v_4dt$. Thus the total displacement of the particle is the same as if the particle moved always

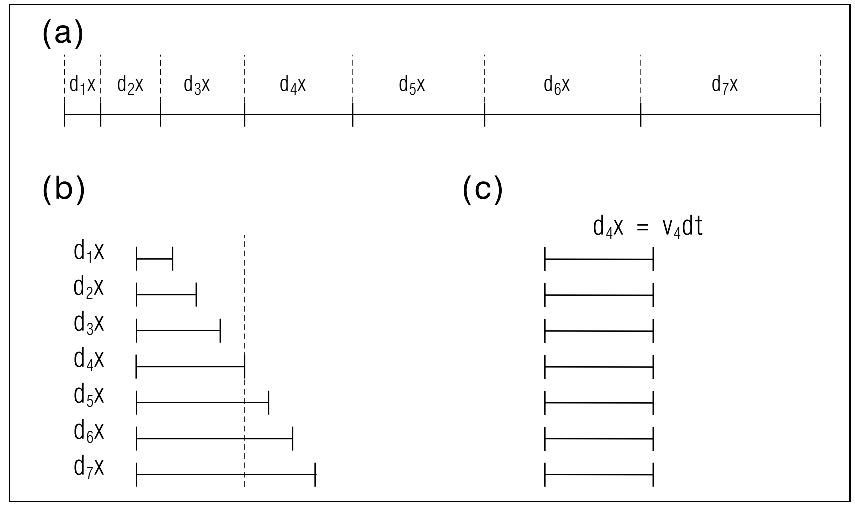


Fig. D-3: Successive small displacements of a particle moving with uniformly increasing velocity. (For clarity, dt has been chosen unrealistically large so that only a few displacements need be shown.) (a) Successive displacements along the line along which the particle moves. (b) Displacements redrawn for easy comparison. (c) Displacements if the particle were traveling with a constant velocity having a value midway between its initial and final velocities.

with a *constant* velocity $v_m = v_4$ having a value *midway* between its initial and final values. Accordingly:

$$\boxed{\Delta x = v_m \Delta t} \qquad \text{(D-5)}$$

Note that the middle value v_m of the velocity is equal to

$$v_m = v_A + \frac{1}{2}(v_B - v_A) = \frac{1}{2}(v_A + v_B) \qquad \text{(D-6)}$$

i.e., it is equal to the average value of the initial and final velocities. Thus our conclusion in Eq. (D-5) can be stated this way:

> If the velocity of a particle changes *uniformly*, the total change of the position coordinate of the particle is equal to the average (or middle) velocity multiplied by the total change of time. (D-7)

Example D-2: Car moving with uniformly changing velocity

While a car is accelerating to pass a truck, the velocity of the car increases uniformly from 20 meter/second to 30 meter/second during a time of 8.0 second. What is the distance traveled by the car during this time?

The average (or middle) value of the velocity during this time is 25 meter/second. Hence the total change of position Δx of the car during this time of 8.0 second is simply (25 meter/sec)(8.0 sec) $= 2.0 \times 10^2$ meter.

The conclusions expressed in Rule (D-4) and Rule (D-7), obtained for the special case where the rate is a velocity are generally valid in the case of any rate. Thus these conclusions can be stated more generally:

If the rate dx/dt is *constant*, the total change Δx is simply equal to the rate multiplied by the total change Δt.	(D-8)

If the rate dx/dt changes *uniformly*, the total change Δx is equal to the *average* (or middle) value of the rate multiplied by the total change Δt.	(D-9)

Finding Functions From Rates (Cap. 4)

D-1 A blood cell moving through a capillary completes its exchange of nutrients with the surrounding tissue in about 2 seconds. If the blood cell moves with a constant velocity of 0.4 millimeter/sec (where 1 millimeter $= 10^{-3}$ meter), what is the distance traveled by the cell during these 2 seconds? (In fact capillaries have approximately this length, just the length needed for adequate exchange between blood and tissue.) *(Answer: 122)*

D-2 A train traveling along a straight section of track approaching a sharp curve must decrease its velocity from 80 mile/hour to 40 mile/hour during a time interval of 0.05 hour. (a) If the train's velocity decreases uniformly, what is the distance it travels during this time interval? (b) To slow the train in this way, at what distance from the curve must the engineer first apply the brakes? *(Answer: 108) ([s-5], [p-8])*

D-3 In 1965 the burning of fossil fuels released carbon into the atmosphere at the approximate rate $R = 4.0 \times 10^9$ ton/year. By 1970 this rate had increased to approximately $R = 5.2 \times 10^9$ ton/year. If this rate R was uniformly changing, what was the total number of tons of carbon released into the atmosphere between 1965 and 1970? *(Answer: 113) ([s-2], [p-9])*

SECT.

E SUMMARY

DEFINITIONS

change; Def. (A-1)

rate; Def. (B-3), Eq. (B-5)

small enough; Def. (B-6)

IMPORTANT RESULTS

Definition of change: Def. (A-1), Rule (B-4)

$\Delta t = t_c - t_0$; $dt = \Delta t$ if Δt is small enough.

Definition of rate: Def. (B-3), Eq. (B-5)

dx/dt

Simple formulas for rates: Rule (C-1), Rule (C-3), Rule (C-9)

If $x = C$, $dx/dt = 0$.
If $x = Ct$, $dx/dt = C$.
If $x = Ct^2$, $dx/dt = 2Ct$.

General properties of changes and rates: Rule (A-3), Rule (A-5); Rule (C-10), Rule (C-11)

If $x = Cu$, $\Delta x = C\Delta u$, $dx/dt = C du/dt$.
If $x = u + w$, $\Delta x = \Delta u + \Delta w$, $dx/dt = du/dt + dw/dt$.

Motion along a straight line: Eq. (D-3), Rule (D-4); Eq. (D-5), Rule (D-7)

If v is constant, $\Delta x = v\Delta t$.
If v changes uniformly, $\Delta x = v_m \Delta t$, where v_m is the average (or middle) velocity.
The same results hold if v denotes *any* rate $v = dx/dt$.

NEW CAPABILITIES

You should have acquired the ability to :

(1) Understand the definition of rate. (Sec. B; [p-1], [p-2], [p-3])

(2) Use a graph of a function to describe and compare values of its rate. (Sec. B; [p-4], [p-5])

(3) Use a simple formula for a function to find an expression or value for its rate. (Sec. C; [p-6], [p-7])

(4) Find total changes in a function which has a constant or uniformly changing rate. (Sec. D; [p-8], [p-9])

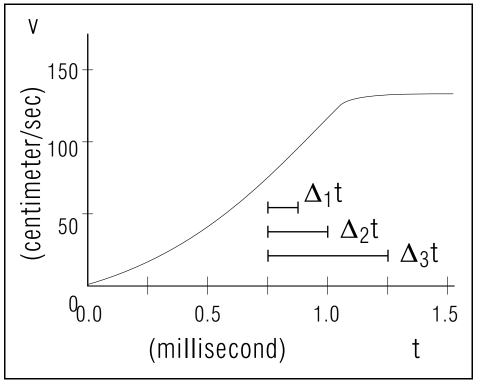


Fig. E-1.

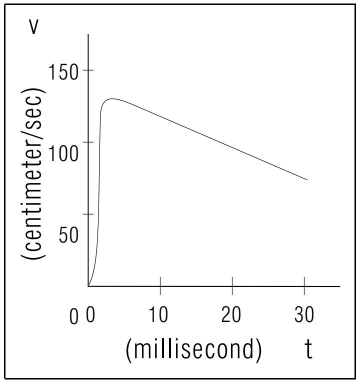


Fig. E-2.

Using the Definition of Rate (Cap. 1 and 2)

E-1 Suppose you know the value of a rate dx/dt for a value t_0, and a slightly larger value t such that $dt = t_c - t_0$ is small enough. Which of the following phrases describe information you could find? (a) Whether the function x is increasing, decreasing, or remaining constant for values of t near t_0. (b) The value x_0 of x corresponding to t . (c) The value of the change dx in x corresponding to dt. (d) Whether the value x_c (corresponding to t_c) is larger, smaller, or the same as x_0. *(Answer: 103)*

E-2 According to recent measurement, the velocity of a jumping flea changes astonishingly quickly, as indicated in Fig. E-1 which shows how the vertical velocity v of a flea changes with time t. (a) Of the time intervals $\Delta_1 t$, $\Delta_2 t$, and $\Delta_3 t$, which are small enough to find the rate $a = dv/dt$ with the best accuracy possible using this graph? (b) For which of the following values of t is the corresponding value of the rate $a = dv/dt$ largest? For which is it smallest? $t = 0.25$ millisecond, $t = 0.75$ millisecond, $t = 1.50$ millisecond. *(Answer: 117)*

E-3 Figure E-2 shows the vertical velocity of the flea during a longer period of time. Which of the following is the closest to the correct value of the rate $a = dv/dt$ for $t = 10$ millisecond? (a) -5 cm/(sec millisecond); (b) -1 cm/(sec millisecond); (c) $-1\,\text{cm/sec}^2$. *(Answer: 121)*

SECT.

F PROBLEMS

Relate Velocity, Distance, and Time

You should be able to use Rule (D-7) describing motion along a straight line, to find values for velocity, distance or time.

F-1 In a 30 mile/hour (or 13 meter/sec) speed zone, a policeman starts his stop watch as a car passes a sign 30 meter from an intersection. The driver of the car, seeing the policeman, immediately decreases his velocity uniformly so that 3.0 sec later, as he arrives at the intersection, his speed is only 5 meter/sec. What is the car's original speed? Was the driver speeding? *(Answer: 107) ([s-4], [p-10])*

Use Graphs to Describe Functions and Rates

F-2 A government economist says, "During the next six months, I expect that the rate of inflation (with time) will decrease, although inflation will continue to increase." Which of the graphs in Fig. F-1 might represent the expected relation between inflation F and time t? *(Answer: 114)*

F-3 For each of the graphs in Fig. F-1, is the function F increasing or decreasing? Is the rate dF/dt increasing or decreasing? *(Answer: 104) (Suggestion: [s-6])*

F-4 In many situations like the following, the rate dF/dt of a function is proportional to the function itself, that is $dF/dt = kF$, where k is a constant quantity. (a) When the number F of organisms (algae, antelopes, people) in a population increases with time, there are more organisms to reproduce, and so dF/dt also increases. Thus $dF/dt = kF$ (approximately) where k is positive. (b) If F is the number of radio-active nuclei in a sample, $dF/dt = kF$ where k is negative, because as nuclei decay and the number F decreases, there are fewer nuclei left which *can* decay. Thus the magnitude of the rate of decay, $|dF/dt|$ also decreases.

A function related to its rate in this way is called an "exponential function." Which of the graphs in Fig. F-1 describes an increasing exponential function F like the one described in part (a)? Which graph shows a decreasing exponential function like the one described in (b)? *(Answer: 110)*

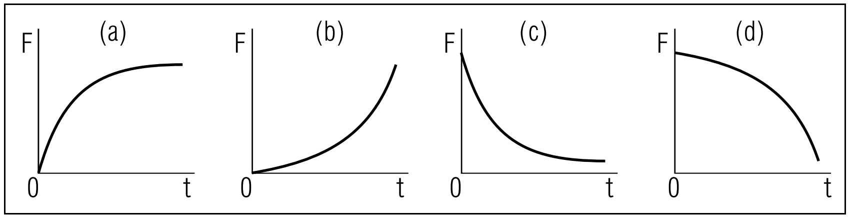
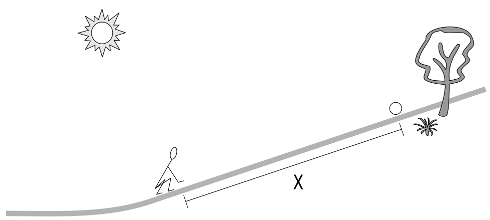


Fig. F-1.

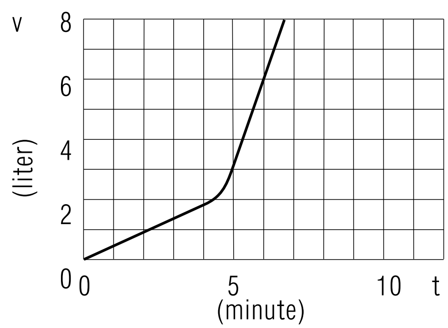
PRACTICE PROBLEMS

p-1 *INTERPRETING SYMBOLS IN THE DEFINITION OF RATE (CAP. 1):* A child throws a ball up a hill, and it rolls back towards him. Thus the distance x between the ball and the child's hand decreases with the time t since the child threw the ball.



At the time 2.1 sec, the value of x is 1.5 meter, and at a slightly later time of 2.3 sec, the value of x is 1.1 meter. If the time interval described is small enough, what is the rate of change of x with respect to t at the time 2.1 sec? *(Answer: 6) Suggestion: Review text problems B-1 and B-3. (Further practice: [p-2])*

p-2 *INTERPRETING SYMBOLS IN THE DEFINITION OF RATE (CAP. 1):* The following graph shows the relation between the time t and the total volume V of oxygen consumed by a man who first lies at rest for four minutes, and then exercises vigorously by running on a treadmill. (Liter is a unit of volume equal to $10^3\,\text{cm}^3$.)

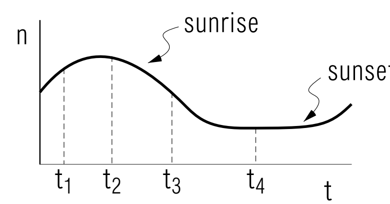


(a) If a time change of 1 minute is small enough, what is the rate dV/dt of the man's oxygen consumption when the time is 2 minute (as the man lies at rest), and when the time is 6 minute (as the man exercises)? *(Answer:*

13) (Suggestion: review text problems B-1 and B-3.))

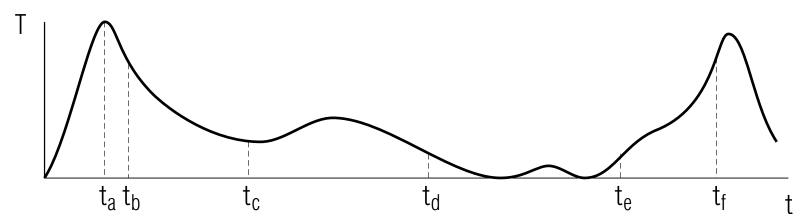
p-3 *RELATING QUANTITIES USING THE DEFINITION OF RATE (CAP. 1):* A blood donor notices that the volume of blood in the collection container increases with a constant rate of $30\,\text{cm}^3/\text{min}$. What time interval is required to fill the $470\,\text{cm}^3$ container, i.e., during what time interval does the volume of blood change by $470\,\text{cm}^3$? Since the rate of change is constant, any time interval is small enough. *(Answer: 16) Now: Return to text problem B-6 and make sure your work is correct.*

p-4 *USING GRAPHS TO DESCRIBE RATES (CAP. 2):* The number n of carbon dioxide molecules present in an air sample varies with time t because photosynthesis (which consumes carbon dioxide) occurs only during daylight. The following graph shows how n varies with t for air samples taken in a forest.



For each of the four values of t indicated on the graph, state whether the corresponding value of the rate $R = dn/dt$ is positive, negative, or zero. *(Answer: 2) Now: Return to text problem B-7 and make sure your work is correct.*

p-5 *USING GRAPHS TO DESCRIBE RATES (CAP. 2):* The following graph shows how the body temperature T of a malaria victim changes with time t.



(a) For each of the values of t indicated on the graph, what is the sign of the rate dT/dt corresponding to this value? (b) In each of the following pairs of values for t, at which value is the corresponding rate dT/dt *larger*: (t_b, t_c), (t_b, t_d), (t_e, t_f)? *(Answer: 12) Now: Return to text problem B-8 and make sure your work is correct.*

p-6 *FINDING RATES FROM SIMPLE FORMULAS (CAP. 3):* What is the rate of change with respect to x of each of the these functions: U, y, and F? Express these rates using the variable x and constant quantities. (a) $U = 7.0\,\text{meter/sec}$ (b) $y = (2.0\,\text{meter}) - (5.0\,\text{meter/sec}^2)x^2$ (c) $F = F_0 + (0.5A)x^2$, where F_0 and A are constant quantities. *(Answer: 7) Now: Return to text problem C-1 and make sure your work is correct.*

p-7 *FINDING RATES FROM SIMPLE FORMULAS (CAP. 3):* Because radioactive chromium atoms decay by emitting detectable particles, these atoms are often substituted for iron atoms in blood cells in order to make those cells identifiable. If there are initially 1.00×10^6 chromium atoms in a blood sample, the number N of such atoms decreases with time t according to this approximate formula:

$$N = (1.00 \times 10^6) - (2.3 \times 10^4\,\text{day}^{-1})t + (3.0 \times 10^2\,\text{day}^{-2})t^2$$

(a) What is the rate of change dN/dt? (b) What is the rate for $t = 0\,\text{day}$ and for $t = 10\,\text{day}$? *(Answer: 14) (Suggestion: Review text problem C-2.)*

p-8 *FINDING FUNCTIONS FROM RATES (CAP. 4):* To estimate the length needed for an airport runway, let us suppose that a plane moves during take-off with uniformly changing velocity, and use the following typical values describing the motion of a large jet. The plane begins its motion with zero velocity, reaching a velocity of 70 meter/sec (and leaving the runway surface) after a time interval of 30 sec. Through what distance does the plane travel during this time interval required to take-off from the runway? *(Answer: 11) (Suggestion: Review text problem D-2.)*

p-9 *FINDING FUNCTIONS FROM RATES (CAP. 4):* Disease of the thyroid gland can be assessed by having a patient drink a solution of radioactive iodine-131, and measuring the time required for this iodine to concentrate in the thyroid gland. The patient undergoing this procedure receives radiation at a rate $R = dN/dt$, where N is the total number of particles emitted by the iodine since the patient drank it. This rate R decreases almost uniformly from an initial value $R_A = 10.7 \times 10^4\,\text{sec}^{-1}$ to

a value $R_B = 9.3 \times 10^4\,\text{sec}^{-1}$ at a time of $1.0\,\text{day} = 8.6 \times 10^4$ sec later. What is the number ΔN of particles emitted by the iodine within the patient during this day? *(Answer: 4) (Suggestion: Review text problem D-3.)*

A More Difficult Problem (Text Sec. F):

p-10 *RELATING VELOCITY, DISTANCE, AND TIME:* After riding his soap-box derby car down a hill, a boy wonders how fast it was going at the bottom. To find out he measures the distance through which the car moves and the time required for the car to reach the bottom of the hill, after beginning with an initial velocity of 0. These quantities are 40 meter and 10 sec, and the car moves with uniformly changing velocity. What is the final velocity of the car at the bottom of the hill? *(Answer: 8) (Suggestion: Review text problem F-1.)*

SUGGESTIONS

s-1 *from Text problem B-3:* We focus attention on a particular value of time, $t_0 = 10\,\text{minute}$, and apply the definition of rate $R = dN/dt$. To interpret these symbols using the information provided, use the relation $dt = t_c - t_0$ to find the changed value t_c. Use the graph to find the corresponding values N_c and N_0, and then the corresponding change $dN = N_c - N_0$. Throughout be careful of signs and units, recalling that a counted number (such as N) has no associated unit.

s-2 *from Text problem D-3:* Suppose x is the total weight (in tons) of carbon released into the atmosphere by fossil fuel burning before a time t. Then we wish to find the change Δx corresponding to the five year time interval Δt between 1965 and 1970. Since the rate dx/dt is uniformly changing, according to statement (D-9) the change Δx is equal to the middle (or average) value of the rate multiplied by the time interval Δt.

s-3 *from Text problem C-2:* First find an expression for dN/dt (in terms of t and numbers with units). Then substitute a value for t into this equation in order to find the corresponding value of dN/dt. Be careful with units, noting that each value of t includes the unit *minute*.

If N is constant, the change dN in N is just zero. Thus the rate dN/dt (when N is constant) is just zero.

s-4 *from Text problem F-1:* Choose symbols for the known and unknown quantities described. Using these symbols, and statement (D-9), write an equation relating the average of the initial and final velocities of the car, the time interval between the initial and final times, and the distance traveled during this time interval. Use this equation to write an expression for the desired quantity, the *initial* velocity of the car.

s-5 *from Text problem D-2:* We wish to find the distance Δx traveled by a train from information about initial and final values for the velocity $v = dx/dt$. Because this velocity is uniformly changing, according to statement (D-9), the distance Δx is just equal to the middle (or average) value of the velocities times the corresponding time interval Δt.

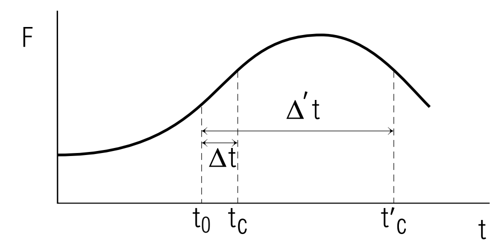
If your answer is 1 mile, you probably used a "middle" velocity 20 mile/hour, which is not the average of 80 and 40 mile/hour.

s-6 *from Text problem F-3:* If a function is represented by a graph, the magnitude of its rate is largest for a value of the variable at which the graph is most steep. Thus if steepness increases with t, then the magnitude $|dF/dt|$ increases with t. But remember, if dF/dt is negative and the magnitude $|dF/dt|$ *increases*, then the rate itself dF/dt becomes more negative and thus *decreases*.

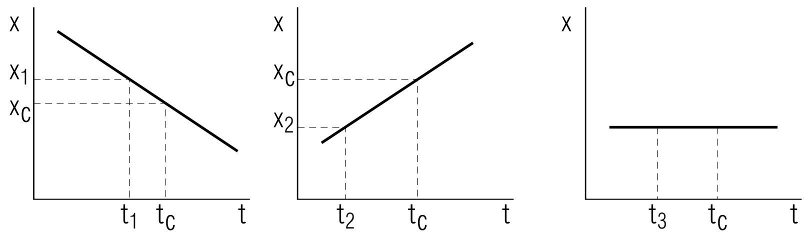
s-7 *from Text problem B-4:* This problem requires knowing about two important aspects of the meaning of the quantities appearing in the definition of rate:

(1) Rate is related to *changes* in a function and in a variable. If only one value of each quantity is available, there is not enough information to find these changes.

(2) A rate is a ratio of *small enough* changes. If a function is described by a graph, a change in the variable (e.g., Δt) is small enough if the graph is approximately straight within that change (that is, straight enough for the precision desired). For example, suppose we wish to find the rate dF/dt at time t_0 for the function F described by the following graph. To find dF/dt with the best precision possible with this graph, the value of $\Delta' t = t'_c - t_0$ shown is *not* small enough, but the value of $\Delta t = t_c - t_0$ is small enough.



s-8 *from Text problem B-7:* Each of the following graphs shows the relation between the function x and variable t in a region sufficiently small that the graph is straight.



For each of the following values of t, first state whether x is increasing, constant, or decreasing near this value. Then state whether the rate dx/dt is positive, zero, or negative for this value of t.

► at t_1: value increasing, constant, decreasing? slope pos, zero, neg?

at t_2: value increasing, constant, decreasing? slope pos, zero, neg?

at t_3: value increasing, constant, decreasing? slope pos, zero, neg?

(Answer: 10) (Note: For further help, review the text discussion of Fig. B-4.) Now: Go to practice problem [p-4].

s-9 *from Text problem B-6:* If it is not obvious how to apply a relation, it often helps to summarize the known and desired information, choosing an algebraic symbol for each quantity. The following chart indicates a convenient choice of symbols for this problem, using P to indicate population, and t to indicate time. Complete this chart by writing the values of known quantities:

►

Description of quantity	*Symbol*	*Value*
original time	t_0	
change in time	dt	
rate at original time	R	
original population	P_0	
change in population	dP	desired
changed population	P_c	desired

Write the definition of rate (using the above symbols):

► ______________________ .

Rearrange the above equation to obtain an expression for the desired quantity dP in terms of symbols for known quantities, and then to find a

value for dP:

► $dP =$ ______________________ $=$ ______________________ .

Use the relation $dP = P_c - P_0$ to find the changed population P_c at the end of the time interval (in 1975):

► $P_c =$ ______________________ .

(Answer: 1) Now: Use this procedure to solve the practice example in practice problem [p-3].

s-10 *from Text problem C-1:* First make sure you can find expressions for the rates of the following simple functions.

For each of the following expressions for x, write the corresponding expression for the rate dx/dt. (Express your answers in terms of numbers and t.)

► If $x = t^2$, then $dx/dt =$ ______________________.

If $x = t$, then $dx/dt =$ ______________________.

If $x =$ any constant quantity, then $dx/dt =$ __________.

Now: Check answer 9 and continue.

Now we review two properties of rates which enable one to find the rates of more complex functions.

(1) Suppose a function x is equal to another function u multiplied by a constant quantity c. Then the rate of x is just the rate of u multiplied by the constant c.

If $x = cu$, then $dx/dt = c(du/dt)$. For each of the following expressions for x, write the corresponding expression for the rate dx/dt.

► If $x = 5t^2$, then $dx/dt =$ ______________________ .

If $x = v_0 t$, where v_0 is a constant quantity, then $dx/dt =$ ______________________ .

(2) Suppose a function x is equal to the sum or difference of two other functions u and w. Then the rate of x is equal to the sum or difference of the rates of u and w:

If $x = u + w$, then $dx/dt = (du/dt) + (dw/dt)$

If $x = u - w$, then $dx/dt = (du/dt) - (dw/dt)$

For each of the following expressions for x, write the corresponding expression for the rate dx/dt.

► If $x = (5\,\text{meter/sec})t - (6\,\text{meter})$, then $dx/dt =$

____________________ .

If $x = at^2 + x_0$, where a and x_0 are constant quantities, then dx/dt

= ____________________ .

(Answer: 3) Now: Go to practice problem [p-6].

s-11 *from Text problem B-8:* To compare values of the rate dT/dx, let us consider separately signs and magnitudes.

(1) For each of the five values of x indicated in Fig. B-8, indicate the sign $(+, -, 0)$ of the corresponding value of dT/dx.

► x_1: __________.

x_2: __________.

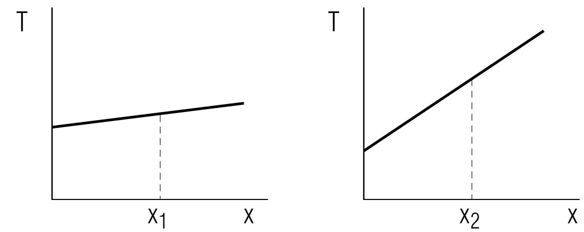
x_3: __________.

x_4: __________.

x_5: __________.

Since all negative values are less than zero, and all positive values larger than zero, this list of signs should help you answer most of the questions in problem B-8.

(2) Next we consider two values of dT/dx which have the same sign. For example, the following graphs show small regions near two values of x, x_1, x_2, for which the rate dT/dx is positive.

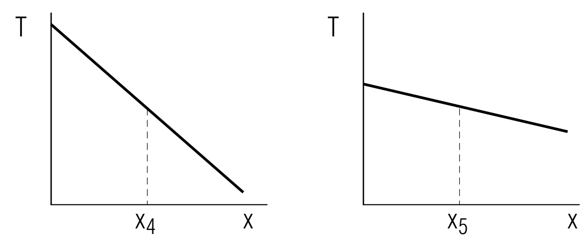


For which of these values of x is the corresponding *magnitude* of the rate dT/dx larger: x_1 or x_2?

For which of these values of x is the corresponding *value* of the rate dT/dx larger: x_1 or x_2?

(Note: For further help, review the text description of Fig. B-4.)

(3) Now consider two values of the variable x, x_4 and x_5 for which the corresponding values of the rate dT/dx are both negative.



For which of these values of x is the corresponding *magnitude* of the rate dT/dx larger: x_4 or x_5?

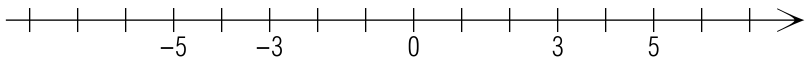
For which of these values of x is the corresponding *value* of the rate dT/dx larger: x_4 or x_5?

(4) Use the word *most* or *least* to complete the following summary of the relation between the appearance of a graph and the rate of-the function it describes.

► The *magnitude* of a rate is largest for a value of the variable near which the graph is ________ steep. The *sign* of the rate (positive, negative, or zero) describes whether the graph is increasing, decreasing, or constant.

(Answer: 5) Now: If right, go to practice problem [p-5]. If wrong, go to Suggestion [s-12].

s-12 *from Suggestion [s-11]:* Let us review the comparison of magnitudes and values for positive and negative numbers. We represent numbers by points on the following line:



First find the magnitudes of each of the numbers indicated on the preceding number line.

► $|3| =$ ______________ , $|5| =$ ______________

$|-3| =$ ______________ , $|-5| =$ ______________

The *magnitude* of a number is indicated by its distance from 0 on the number line. Thus 5 and −5 both have magnitude 5, and $|-5|$ is *larger* than $|-3|$. However, the value of a number is indicated by its position on the number line, a larger number lying to the right of a smaller number.

Which number has the larger *value*, $|-3|$ or $|-5|$?

► ________________

(Answer: 15) Now: Return to Suggestion [s-11] and correct your work.

ANSWERS TO PROBLEMS

1. $t_0 = 1970\,\text{year}$, $dt = 5\,\text{year}$, $R = 2.4 \times 10^6\,\text{year}^{-1}$
   $P_0 = 2.03 \times 10^8$; $R = dP/dt$, $dP = R(dt) = 1.2 \times 10^7$
   $P_c = P_0 + dP = 2.15 \times 10^8$
2. t_1: +; t_2: 0; t_3 : −; t_4: 0
3. (1) $10t$, v_0 (2) 5 meter/sec, $2at$
4. $\Delta N = 8.6 \times 10^9$
5. (1) x_1: +; x_2: +; x_3: 0; x_4 : −; x_5 : −
   (2) x_2, x_2 (3) x_4, x_5 (4) most
6. −2 meter/sec (Check that sign and unit are correct.)
7. a. $dU/dx = 0$
   b. $dy/dx = (-10\,\text{meter/sec}^2)x$
   c. $dF/dx = Ax$
8. 8 meter/sec
9. $2t$, 1, 0
10. t_1: decreasing, negative; t_2: increasing, positive;
    t3: constant, zero
11. 1.0 or 1.1×10^3 meter (1050 meter precise to two significant figures)
12. a. t_a: 0; t_b : −; t_c: 0 ; t_d : −; t_e: +; t_f: +
    b. t_c, t_d, t_f
13. For $t = 2$ minute, $dV/dt = 0.5$ liter/minute. For $t = 6$ minute, $dV/dt = 3$ liter/minute.
14. a. $dN/dt = (-2.3 \times 10^4\,\text{day}^{-1}) + (6.0 \times 10^2\,\text{day}^{-2})t$
    b. For $t = 0$ day, $dN/dt = -2.3 \times 10^4\,\text{day}^{-1}$. For $t = 10$ day, $dN/dt = -1.7 \times 10^4\,\text{day}^{-1}$
15. $|3| = 3$, $|5| = 5$, $|-3| = 3$, $|-5| = 5$; −3 is larger than −5.
16. 16 minute (15.7, precise to two significant figures)

101. a. $R = dy/dx$

b. $R = dy/dx = 0.5$

102. a. $dx/dt = (10\,\text{meter/sec}^2)t$

b. $dy/dt = 0$

c. $ds/dt = a_0 t + v_0$

103. a., c., and d. can be found

104. a. F: increasing, dF/dt decreasing;

b. F: increasing; dF/dt: increasing;

c. F: decreasing, dF/dt: increasing

d. F: decreasing; dF/dt: decreasing.

105. a. $t_c = 4\,\text{day}$, $N_0 = 200$, $N_c = 400$, $\Delta N = 200$

b. $t_c = 6\,\text{day}$, $N_0 = 450$, $N_c = 475$, $\Delta N = 25$

106.

t	x	dx/dt
2 month	negative	zero
4 month	zero	positive
10 month	positive	negative

107. 15 meter/sec; yes

108. a. 3 mile

b. 3 mile

109. $-20\,\text{minute}^{-1}$ or $-20(1/\,\text{minute})$ (Check that sign and unit are correct.)

110. graph b, graph c

111. a. $dN/dt = (3.0 \times 10^{18}\,\text{minute}^{-1}) - (1.2 \times 10^{17}\,\text{minute}^{-2})t$

b. For $t = 0$ minute, $dN/dt = 3.0 \times 10^{18}\,\text{minute}^{-1}$. For $t = 10$ minute, $dN/dt = 1.8 \times 10^{18}\,\text{minute}^{-1}$ (Check units carefully.)

c. $dN/dt = 0$

112. $\Delta B = 2.5\,\text{foot}$

113. 2.3×10^{10} ton

114. a.

115. a. dT/dh

b. positive, negative, or zero

c. (degree Centigrade)/meter

116. a. x_2

b. x_2

c. x_3

d. x_5

e. x_1

117. a. Both $\Delta_1 t$ and $\Delta_2 t$ are small enough.

b. Largest at 0.75 millisecond. Smallest at 1.50 millisecond.

118. a. 20 meter

b. 4 meter

c. 20 meter/sec

d. t_4

119. a. 1.2×10^7

b. 2.15×10^8

120. a. Cannot be found, because from one value of T (or h) cannot find a change dT (or dh).

b. $dv/dt = [(130 - 120)\,\text{cm/sec}]/[(0.15 - 0.10)\,\text{sec}] = 2 \times 10^2\,\text{cm/sec}^2$

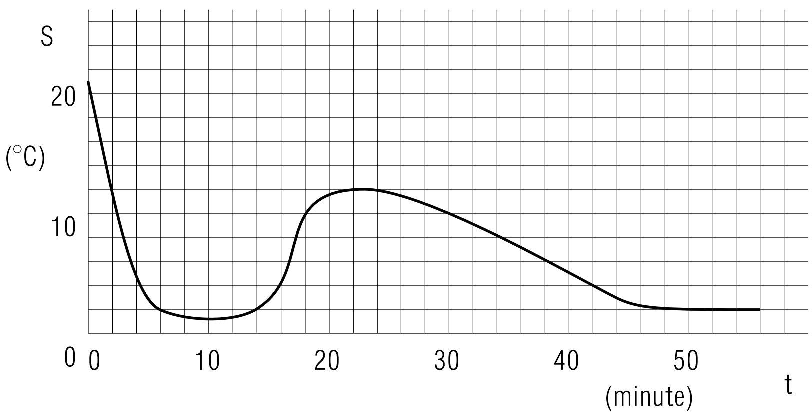
121. b.

122. 0.8 millimeter $= 8 \times 10^{-4}$ meter.

MODEL EXAM

1. **Temperature of fingers immersed in ice water.**

   When fingers are immersed in ice water, the skin temperature S changes as a function of time t as described by the following graph. The temperature S is measured in terms of the unit degree centigrade (abbreviated °C).



   a. For which of the following values of t is the corresponding value of S *positive* and the corresponding value of dS/dt *zero*? 4 minute, 6 minute, 20 minute, and 24 minute.

   b. What is the value of the rate dS/dt for $t = 30$ minute?

2. **Motion of a car.**

   A car passes a service station and thereafter moves so that the following equation relates its distance x from the service station to the time t since passing the station.

$$x = (20\,\text{meter/sec})t + (0.50\,\text{meter/sec}^2)t^2$$

   a. Write an expression for the rate (velocity) $v = dx/dt$ in terms of numbers (with units) and the variable t.

   b. What is the car's velocity 10 sec after passing the service station?

   Well beyond the service station, the car runs out of gas. At the instant it runs out of gas the car's speed is 20 meter/sec, and thereafter its speed decreases uniformly, until, after a time interval of 100 sec, the car comes to rest.

   c. What distance does the car travel during this time interval of 100 sec?

   d. Suppose the car had not run out of gas, but had continued to travel with its original speed along the straight road. What distance would the car have traveled during the same time interval of 100 sec?

Brief Answers:

1. a. 24 minute

   b. between −0.4 and −0.6 °C/ minute

2. a. $dx/dt = (20\,\text{meter/sec}) + (1.0\,\text{meter/sec}^2)\,t$

   b. $dx/dt = 30\,\text{meter/sec}$

   c. 1000 or 1.0×10^3 meter

   d. 2000 or 2.0×10^3 meter