PHYSNET

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TIME-DEPENDENT PERTURBATIONS I

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TIME-DEPENDENT PERTURBATIONS I

by

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Title: **Time-Dependent Perturbations I**

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Input Skills:

1. Vocabulary: coupled differential equations, potential energy (of a charged particle in a uniform electric field).
2. Unknown: assume (MISN-0-390).

Output Skills (Knowledge):

K1. Given a Hamiltonian of the form $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'(t)$, where $\mathcal{H}'(t)$ is a small time-dependent perturbation and complete sets of eigenfunctions and eigenvalues, $\{\phi_n\}$ and $\{E_n\}$ of $\mathcal{H}_0$:

(a) write $\psi(x,t)$ as a time-dependent linear combination of the ϕ_n.

(b) Substitute (a) into the S-equation and obtain an exact set of coupled differential equations for the coefficients of the expansion.

(c) Assuming that the system was initially in an eigenstate of $\mathcal{H}_0$, solve the equations of (b) to first order in the perturbation $\mathcal{H}$.

K2. Assuming $\mathcal{H}'$ has no explicit time-dependence, derive the transition probabilities of the system under the influence of the perturbation and discuss its implications in the manner of Saxon pages 210-211. Obtain the Golden Rule by generalizing the result of K2 to where the transition is made to any member of a continuous collection of states, all of whose energies are close to the initial energy. State the circumstances under which this result is valid. Show how this result leads to the familiar exponential decay law.

Output Skills (Problem Solving):

S1. Apply the results of K2 to solve problems of the type given in the procedures.

External Resources (Required):

1. D. S. Saxon, *Elementary Quantum Mechanics*, Holden-Day, Inc. (1968).

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1. Introduction

We turn now to the important task of dealing with perturbations which depend upon time. Sometimes the time-dependence lies only in the turning on and off of the disturbing force, as in the case of an atom lying in an external magnetic field which can be switched on and off by the experimenter. At other times the perturbation has *explicit time dependence*, as in the case of electromagnetic radiation incident on an atom - if the radiation is monochromatic, the electric field associated with it varies sinusoidally in time.

In this unit we shall consider only the first kind of time-dependence and go on to treat the second kind in the next unit. Among our objectives is the derivation of Fermi's "Golden Rule" which is of great value for a multitude of practical calculations.

2. Procedures

The material for this unit is in Saxon pages 208-213. This material is available in the PA library. Ask for it as "the readings for CBI Unit 391."

There will be no references to Anderson in this unit. All references refer to Saxon.

1.(a) Read section VII. 6 of Saxon through equation 60. Equation 60 is the desired expansion.

(b) Continue reading through equation 61. This equation is the desired set of coupled equations. Fill in all the steps leading to this result. Note the definition $H'_{mn}(t) \equiv \int_{-\infty}^{\infty} \phi_m^*(x)(\mathcal{H}'(t)_{\phi_n}(x))dx$. The use of only one spatial variable x instead of $\vec{r}$ is done for brevity only; the results apply equally well to 3-dimensional wave-functions with x replaced by $\vec{r}$.

(c) Continue reading through the results for c_k and c_m (equation 62 and the equation above it). Fill in all steps leading to these results.

2. Continue reading through equation 63 and fill in all missing steps. This is the desired result. Note the graph of the transition probability on page 211. About what energy is the graph peaked? This corresponds to conservation of energy; the perturbation is small and does not greatly change the energy of the system.

Note that only 1 matrix element of $\mathcal{H}'$ appears in the results for c_k and c_m. Thus these results are called the *first order* perturbation theory results. Successively better approximations to the solutions of equation 61 will involve higher and higher powers of $\mathcal{H}'$; if $\mathcal{H}'$ is sufficiently small, these higher terms can be neglected. Here then is the fundamental approach of time-dependent perturbation theory: Expand the state of the system as a power series in the perturbation (also called the "interaction") $\mathcal{H}'$. If $\mathcal{H}'$ is sufficiently small, only the first few terms will be needed.

This doesn't always work but it is often the only sensible approach to a complicated Hamiltonian.

Now read the discussion following equation 63 through the end of the paragraph surrounding figure 3. Skip the remarks on why the rapid increase with time might be expected; we have not covered the necessary background in degenerate state time-independent perturbation theory. Make sure you can answer the following questions.

(a) Assuming E_k is much different from E_m, what can you say about $|c_m(t)|^2$ if $\mathcal{H}$ is small?

(b) Suppose $E_m \approx E_k$. How does $|c_m(t)|^2$ vary with time? Why is this time-dependence surprising? Under what circumstances will the expected time-dependence be realized? item [(c)] How does width of the graph of $|c_m(t)|^2$ vs. E_m depend on t? Is this behavior reminiscent of the uncertainty principle? This is not quite a direct application of the uncertainty principle. Why?

3. Continue reading through equation 67. Equation 67 is called the "Golden Rule." Fill in all the steps leading to this result.

If there are N systems in the state k, WN gives the number of transitions per unit time from the state k to the state m. (Think of N nuclei in excited state k decaying by α-emission to final state m).

Exercise: If there are N systems in the state k at time t, how many are there at time $t + dt$? Assume that m is the only state to which the perturbation induces transitions. Use this result to show that

$N(t) = N(0)e^{-Wt/\hbar}$. What is the mean life-time of the state k? How should these results be modified if the perturbation connects k to more than one state?

Now read the discussion following equation 67 through the end of the paragraph preceding the one containing equation 71. Under what conditions is the Golden Rule valid?

4. To gain practice with these ideas, solve problem 15a and the following problem:

   Let perturbation of problem 15a act on an electron (mass m, charge e) in an infinite one-dimensional square well. If the electron is initially in its ground state, to what excited states can the perturbation induce a transition? What is the probability of finding the electron in the 2nd, 3rd, and 4th excited states at the time τ, if the perturbation is turned on at $t = 0$?

   (To do these problems you will need the potential energy function corresponding to an electron in a uniform electric field. What is this potential? Use this potential as the perturbation $\mathcal{H}$ in the results of Output Skill K2.)

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