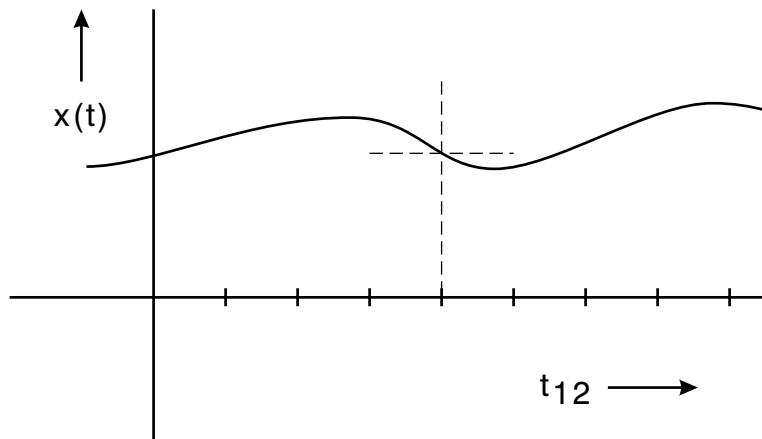


COMPUTER ALGORITHM FOR THE DAMPED DRIVEN OSCILLATOR



COMPUTER ALGORITHM FOR THE DAMPED DRIVEN OSCILLATOR

by
Peter Signell

1. Derivation	
a. Equation for a Damped Driven Oscillator	1
b. Approximate Derivatives by Finite Differences	1
c. Net-Point Times	2
2. Application	
a. Iterating the Recurrence Relation	3
b. Starting the Iteration	4
Acknowledgments	4

Title: **Computer Algorithm for the Damped Driven Oscillator**

Author: Peter Signell, Dept. of Physics, Mich. State Univ

Version: 2/1/2000

Evaluation: Stage 0

Length: 1 hr; 12 pages

Input Skills:

1. Expand a given function about a given point using Taylor's series (MISN-0-4).

Output Skills (Knowledge):

- K1. Given the force acting on a damped driven oscillator along with the oscillator's position and velocity at a specified time, derive a Numerov type algorithm for the approximate numerical calculation of the oscillator's position at all past and future times.

Post-Options:

1. "Response of a Damped Driven Oscillator" (MISN-0-30).
2. "Damped Driven Oscillations; Mechanical Resonances" (MISN-0-31).
3. "Laplace Transform Solution for the Damped Driven Oscillator" (MISN-0-47).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

PROJECT STAFF

Andrew Schnepf	Webmaster
Eugene Kales	Graphics
Peter Signell	Project Director

ADVISORY COMMITTEE

D. Alan Bromley	Yale University
E. Leonard Jossem	The Ohio State University
A. A. Strassenburg	S. U. N. Y., Stony Brook

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

© 2001, Peter Signell for Project PHYSNET, Physics-Astronomy Bldg., Mich. State Univ., E. Lansing, MI 48824; (517) 355-3784. For our liberal use policies see:

<http://www.physnet.org/home/modules/license.html>.

COMPUTER ALGORITHM FOR THE DAMPED DRIVEN OSCILLATOR

by
Peter Signell

1. Derivation

1a. Equation for a Damped Driven Oscillator. The equation to be solved is that of the damped harmonically-driven oscillator¹ acted on by the force:

$$F(t) = -kx(t) - \lambda v(t) + F_0 \cos(\omega t).$$

Since²

$$F(t) = ma(t) = m \frac{dx(t)}{dt} \equiv mx''(t),$$

and

$$v(t) = \frac{dx(t)}{dt} \equiv x'(t),$$

our equation to be solved is:

$$mx''(t) + kx(t) + \lambda x'(t) - f(t) = 0, \quad (1)$$

where

$$f(t) \equiv F_0 \cos(\omega t).$$

1b. Approximate Derivatives by Finite Differences. We now make two power series expansions³ about time t :

$$x(t + \Delta) = x(t) + \Delta x'(t) + (\Delta^2/2)x''(t) + (\Delta^3/6)x'''(t) + \dots$$

$$x(t - \Delta) = x(t) - \Delta x'(t) + (\Delta^2/2)x''(t) - (\Delta^3/6)x'''(t) + \dots$$

We will choose Δ sufficiently small so that we can disregard all terms after Δ^2 without incurring much error. Then add and subtract the above equations to obtain:

$$x(t + \Delta) + x(t - \Delta) \simeq 2x(t) + \Delta^2 x''(t)$$

¹See "Damped Driven Oscillations; Mechanical Resonances" (MISN-0-31).

²See "Particle Dynamics - The Laws of Motion" (MISN-0-14).

³See "Taylor's Series for the Expansion of a Function About a Point" (MISN-0-4).

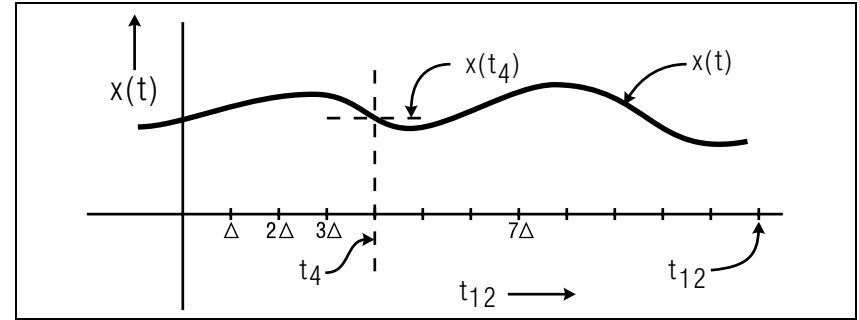


Figure 1. The labeling system for time net-points.

$$x(t + \Delta) - x(t - \Delta) \simeq 2\Delta x'(t)$$

or

$$x''(t) \simeq \frac{1}{\Delta^2} [x(t + \Delta) - 2x(t) + x(t - \Delta)]$$

$$x'(t) \simeq \frac{1}{2\Delta} [x(t + \Delta) - x(t - \Delta)]$$

which are often quoted in calculus courses.⁴

1c. Net-Point Times. We need a more succinct labeling system at this point or we won't be able to see the forest for the trees. We define discrete "net-point" times as: $t_n \equiv n\Delta$, where Δ is some small time interval (see Fig. 1). We can then rewrite the derivatives at time t_n as:

$$x''_n \simeq \frac{1}{\Delta^2} (x_{n+1} - 2x_n + x_{n-1}) \quad (2)$$

$$x'_n \simeq \frac{1}{2\Delta} (x_{n+1} - x_{n-1}) \quad (3)$$

Then at time t_n our basic Eq. (1) becomes:

$$mx''_n + kx_n + \lambda x'_n - f_n = 0.$$

Substituting for x''_n and x'_n :

$$\frac{m}{\Delta^2} \left(x_{n+1} - 2x_n + x_{n-1} \right) + kx_n + \frac{\lambda}{2\Delta} (x_{n+1} - x_{n-1}) - f_n = 0$$

⁴The second equation above is used in "Response of a Damped Driven Oscillator" (MISN-0-30) where a microcomputer is utilized in finding solutions.

Collecting coefficients of the x values, we get:

$$x_{n+1} = Ax_n + Bx_{n-1} + \Delta^2 C_n \quad (4)$$

where:

$$\begin{aligned} A &= \left(2 - \Delta^2 \frac{k}{m}\right) / D \\ B &= -\left(1 - \Delta \frac{\lambda}{2m}\right) / D \\ C_n &= \left(\frac{f_n}{m}\right) / D \\ D &= 1 + \Delta \frac{\lambda}{2m} \end{aligned}$$

2. Application

2a. Iterating the Recurrence Relation. Equation (4) is called a three point recurrence relation because it relates x at three successive time net-points. If you know x_0 and x_1 , you can use this relation to calculate x_2 . Then knowing x_1 and x_2 , you can calculate x_3 , etc. You can keep up this recurrence procedure until you have reached the time for which you wish to know the solution. For example, this might be a time sufficiently large so that transient effects have died away and only the steady state solution remains.

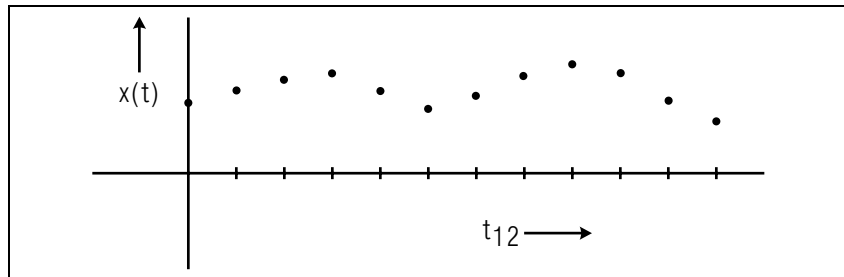


Figure 2. The Calculated Curve. Draw a smooth curve through the plotted points and you have $x(t)$.

2b. Starting the Iteration. With a three-point recurrence relation one can calculate values at successive time net-points, but one needs two adjacent values in order to get started. In the example cited in the preceding paragraph, one needed x_0 and x_1 in order to start the procedure. Those two initial values generally come from a specification of a “complete set of initial conditions.” Our Eq. (1), being a second order equation for x , always requires that one specify two independent conditions on x .

A common case is where both the position and velocity are known at some instant of time. Here we will call that time $t = 0$; hence we know x_0 and x_1 by solving Eq. (3) at $n = 0$ for x_{-1} (Note: $v_0 = x'_0$), and using that to eliminate x_{-1} from Eq. (4) for $n = 0$. We thus obtain x_1 in terms of x_0 and v_0 (usually given quantities):

$$x_1 = \left(1 - \Delta^2 \frac{k}{2m}\right) x_0 + \left(\Delta - \Delta^2 \frac{\lambda}{2m}\right) v_0 + \Delta^2 \frac{f_0}{2m}.$$

Now the algorithm is complete. Put in x_0 and v_0 to get x_0 and x_1 , x_0 and x_1 to get x_2 , x_1 and x_2 to get x_3 , etc.

▷ How would you obtain position values for times earlier than the one for which x and v are initially known? Answer: To see the answer, replace each of the following letters by its successor in the alphabet: mdfzshud cdksz.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

MODEL EXAM

1. The force acting on a damped driven oscillator is $F = -kx - \lambda v + F_0 \cos(\omega t)$; the oscillator's initial position and speed are x_0 and v_0 . Derive without notes, a complete Numerov-type algorithm for the (approximate) calculation of the oscillator's position at all times. NOTE: The algorithm is to include the method of use of any three point recurrence relation such as:

$$x_{n+1} = Ax_n + Bx_{n-1} + \Delta^2 C_n$$

where:

$$\begin{aligned} A &= \left(2 - \Delta^2 \frac{k}{m}\right) / D \\ B &= -\left(1 - \Delta \frac{\lambda}{2m}\right) / D \\ C_n &= \left(\frac{f_n}{m}\right) / D \\ D &= 1 + \Delta \frac{\lambda}{2m} \end{aligned}$$

and boundary conditions such as:

$$x_1 = \left(1 - \Delta^2 \frac{k}{2m}\right) x_0 + \left(\Delta - \Delta^2 \frac{\lambda}{2m}\right) v_0 + \Delta^2 \frac{f_0}{2m}.$$

Brief Answers:

1. See this module's *text*.

