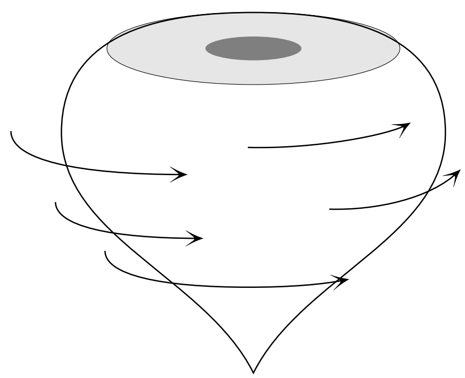
PHYSNET

MISN-0-36

ROTATIONAL MOTION OF A RIGID BODY



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

ROTATIONAL MOTION OF A RIGID BODY

by

J. S. Kovacs

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Input Skills:

1. Define moment of inertia (MISN-0-35).
2. Explain: principal axes of inertia, principal moment of inertia (MISN-0-35).
3. Describe the consequences of an external torque on the angular momentum of an extended object (MISN-0-34).
4. Identify all of the external forces acting on an extended object (MISN-0-15).
5. Apply Newton's Second Law to determine the acceleration of objects under the action of given forces (MISN-0-16).
6. State the Principal Axis (Steiner's) Theorem (MISN-0-35).
7. Apply Newton's second law to determine the instantaneous motion of the center of mass of an extended object (MISN-0-16).
8. Write the expression for the point-mass kinetic energy (MISN-0-41).
9. Determine the linear velocity of a point on a object rotating with a given angular velocity (MISN-0-9).

Output Skills (Problem Solving):

S1. Solve problems using the equation of motion of a rotating rigid body when the motion is about any fixed axis, as well as when the motion is abut a principal axis.

S2. Calculate the kinetic energy of rotation of a rotating rigid body and use this as an additional form of kinetic energy in solving problems using the conservation of energy.

External Resources (Required):

1. M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For availability, see this module's *Local Guide*.

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1. Introduction

Newton's second law, $\vec{F} = m\vec{a}$, where $\vec{F}$ is the resultant of the external forces exerted), when applied to an object that has finite dimension (i.e., not a point object) determines the acceleration of the center of mass of that object. Successive integrations then yield the results usually sought after: the description of the motion of the center of mass, its position given as a function of time. Such an object's motion, however, may involve more than just the motion of its center of mass. The various parts of the object may move relative to the center of mass. In this unit the rotations of a rigid body as a whole about some fixed axis are investigated. Rotation about some axis through the center of mass, will constitute the complete description of the motion of such objects.

2. About a Principal Axis, Fixed In Space

When an object rotates about one of its principal axes the angular momentum vector and the angular velocity vector are parallel[1] and related to one another by a constant of proportionality which is determined by the distribution of mass of the object with respect to the axis.[2] The constant is the *moment of inertia* with respect to this principal axis:

$$\vec{L} = I\vec{\omega} \tag{1}$$

Unless there are external torques disturbing this rotating object (such as perhaps the torque due to friction which may also slow the rotating object to a stop), the vector $\vec{L}$ remains constant in magnitude and direction: *Angular momentum is conserved.* When there are external torques, the *net external* torque can be related to the resulting time rate of change of

[1]See "Calculation of Moments of Inertia, Principal Axes" (MISN-0-35), where the distinction is made between the rotation of an object about any axis and the rotation about one of the *principal axes.*

[2]It actually is the second moment of the distribution, so-called because each mass element is weighted by the *square* of its distance from the axis of rotation in determining its contribution to the total I: $I = \int r^2 dm$.

angular momentum,

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \tag{2}$$

as can be derived directly from Newton's laws.[3] The vector $\vec{\tau}$, in general, need not be in the same direction as $\vec{L}$, so that $d\vec{L}/dt$ may be in a different direction than $\vec{L}$. This results in an $\vec{L}$ which changes its direction with time, as illustrated by the gyroscope.

If, however, the axis of rotation happens to be fixed so that the external torque is also along the same axis as $\vec{L}$ and $\vec{\omega}$, then $d\vec{L}/dt$ is also along this direction and the change in $\vec{L}$ is in magnitude only. Furthermore, with the axis fixed (in space as well as in the body), the moment of inertia remains constant, so that we have

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} \tag{3}$$

where $\vec{\alpha}$, the angular acceleration vector, is along the same axis as all the other vectors.[4] With these restrictions, there is a close parallel between $\vec{F} = M\vec{a}$ and $\vec{\tau} = I\vec{\alpha}$. The first of these relates the *net external force* on an object to the resulting acceleration of the center of mass of that object. ($\vec{F}$ is the "cause" that produces the "effect": the acceleration of the object's center of mass.) The second one, $\vec{\tau} = I\vec{\alpha}$, relates the external torque to the resulting angular acceleration. The quantity I plays the analogous role in rotational motion to the role that M plays in translational motion.[5]

As an illustration, consider the rotation of the earth. The direction of the axis of rotation is not quite fixed in space. However, for most considerations it may be viewed as fixed. Assuming this, how much external torque must be applied for one day to increase the earth's rotation such that the length of the day is decreased to 23 hours? Applying the above relation, Eq. (3), and assuming that a constant torque is applied for one day, the answer is 8.52×10^{28} newton-meters (assuming the earth to be

[3]See "Torque and Angular Momentum in Circular Motion" (MISN-0-34) for the basis of this relationship.

[4]Even if the axis of rotation is *not* a principal axis, as long as it is fixed and as long as the net external torque is along this axis, an equation related to the equation derived below is applicable. That is, Eq. (3) applied only to the along-the-axis components of the vector quantities appearing in it would be correct even if the axis is not a principal one.

[5]Cases where translational and rotational motion occur simultaneously are dealt with in "Translational Plus Rotational Motion of a Rigid Body" (MISN-0-43).

a sphere of mass 5.98×10^{24} kilograms and of radius 6.37×10^{6} meters). If this torque were produced by a tangential force applied at the equator the magnitude of the force would be 1.34×10^{22} newtons.

3. About a Fixed Axis Not Through CM

Consider the motion of a baseball bat in the hand of a batter. Neglecting for the time the important effect of the motion of the batter's arms and his body in general, the motion of his bat can be viewed ideally as a rotation about an axis near the thin end of the bat, perpendicular to the bat. This rotation is clearly not about an axis through the center of mass of the bat, which is considerably nearer to the "fat end" of the bat.

If such a bat were held stationary in the conventional position by a batter, before the pitched ball struck it, the torque produced by the impact of the ball would tend to rotate the bat. (If the bat were loosely held, the ball would "knock the bat out of the batter's hands.") If this torque were known, the angular acceleration of the bat's rotation could be determined by the method described in Section 2. The batter in attempting to provide a stationary pivot around which the bat rotates would have to exert a force to keep the bat from being knocked from his hands. How much force must he exert? The answer comes from the direct application of Newton's law to the motion of the center of mass of the bat. That is, *the instantaneous acceleration of the center of mass of the object is determined by the net external force acting on it at that instant.*

Consider the situation illustrated in Fig. 1. The ball is exerting a force F at the point of impact A. With the point B held fixed, the bat tends to rotate around point B with an angular acceleration determined by the torque about B produced by F. From this known angular acceleration, the *tangential acceleration*[6] of any point on the bat can be determined. In particular, the instantaneous linear acceleration of the center of mass can be determined, which in turn determines the net force on the bat. Knowledge of the force of impact then determines the force with which the bat must be held.[7]

[6]See "Torque and Angular Acceleration for Rigid Planar Objects: Flywheels" (MISN-0-33) for the relationship between angular acceleration and linear acceleration when the path of the motion is circular (in which case the *linear acceleration* is *tangent* to the circle, hence the alternative term *tangential acceleration.*

[7]The reaction force to this force, namely the force exerted on the batter's hand by the bat, is responsible for the familiar "sting" felt by batters when the bat is not held in the proper place relative to the point of impact with the ball.

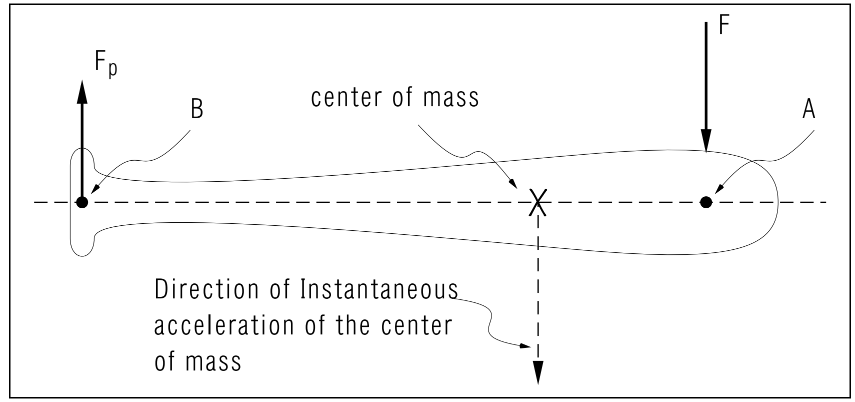

Figure 1. A bat held stationary at point B must have a force F_p exerted (at the instant of impact) in the direction shown if the impact force F exerted at point A will result in a pure rotation of the bat about a fixed axis through B.

With the point of impact A located a distance A from the center of mass and the bat held fixed at B a distance B from the center of mass, the force that the batter must exert on the bat, F_p (shown in Figure 1 directed opposite to F) is,

$$F_p = \left[\frac{K^2 - AB}{K^2 + B^2}\right] F, \tag{4}$$

where K is the radius of gyration of the bat relative to an axis through the center of mass. This indicated that F_p can be zero: the batter need not exert a force on the bat and conversely, the bat won't exert a force on his hand, if the point of impact and the point where the bat is held are such that the product of their distances from the center of mass equals the square of the radius of gyration relative to the center of mass.

It should be noted that this above result, Eq. (4) with F_p directed as shown in Figure 1, represents the total force needed at the axis of rotation only if the object (the bat) has not acquired any angular velocity— before the bat has rotated appreciably. When the bat has begun to rotate, an additional force must be provided to keep the point B stationary. This is the *centripetal force*,[8] $MB\omega^2$, directed to the left in Figure 1 along the line from the point B to the center of mass. This is just the force

[8]See MISN-0-17 for the development of and illustration of centripetal force.

that must be provided to keep an object (the center of mass in this case) moving in a circular path.

4. Rotational Kinetic Energy

The kinetic energy of a point-like mass moving with velocity v is given by the expression $\frac{1}{2}Mv^2$. This is appropriate even when the point-like mass is a part of an extended object which is rotating. With the object rotating about some axis with angular velocity ω, the kinetic energy of a point-like mass located a distance R from the axis is,

$$E_k = \frac{1}{2}M(\omega R)^2 = \frac{1}{2}(MR^2)\,\omega^2. \qquad (5)$$

The *total kinetic energy* of the extended object as a whole is just the sum of a large number of such terms, one for each of the point-like masses that make up the object. This sum for a continuous mass distribution must be expressed as an integral over all elements of the object. However, in this integral or sum, ω has the same value for each element of mass, irrespective of its mass and distance from the axes. The integral is then only over the "MR^2" contributions of the object. But this is the moment of inertia of the object relative to the axis about which the rotation occurs:

$$I = \sum_i M_i R_i^2 \rightarrow \int R^2 dM \qquad \text{(for continuous distributions).} \qquad (6)$$

Thus, the total rotational kinetic energy of an object undergoing rotation about some fixed axis is:

$$E_k(\text{rotation}) = \frac{1}{2}I\omega^2. \qquad (7)$$

If the axis is not fixed, then the velocity of each point-like mass is not related to ω in the simple way that it is for the case of pure rotation. The consequence of this is that the total kinetic energy of the object includes terms in addition to what there is in Eq. (7) above. These additional terms are easily interpretable, however.

As an example, consider a spherical ball rolling without slipping on a flat surface. A bit of reflection will verify that because the ball is not slipping, if its center is moving forward with a speed V, then the angular velocity it has about an axis through its center is $\omega = V/R$ where R is the radius of the ball. (If it slips ω will have a value greater than this.) The

ball has rotational kinetic energy given by Eq. (7) above. It would have this rotational kinetic energy even if it were spinning in place (if it spun with the same angular velocity, ω). Furthermore, if the ball didn't rotate at all while it slid along the horizontal surface with speed V, its kinetic energy would be given by the usual expression for the translational kinetic energy of an object: $MV^2/2$. When these two motions are combined, as in the case of rolling without slipping, it should be no surprise that the total kinetic energy is the combined kinetic energy of each of these separate cases of motion:

$$E_k = \frac{1}{2}I\omega^2 + \frac{1}{2}MV^2,$$

Where the first term is the kinetic energy of rotation about an axis through the center of mass while the second term is the kinetic energy associated with the translational motion of the center of mass. The effect of this result is illustrated by considering the following demonstration: If a sphere of mass M is placed on an:incline which is such that the mass starts out at rest at a vertical. height H above the bottom of the incline, then if this mass slid (without rolling and without frictional loss) down the incline, it would have a speed $\sqrt{2\,gH}$ when it got to the bottom. This result is immediately obtained by observing that the potential energy the sphere had at the top of the incline is converted to kinetic at the bottom. If, however, this sphere rolled without slipping down the incline, the total energy it would acquire at the bottom would be the same as in the slipping case, but it would be divided into rotational and translational kinetic energy. Consequently, its translational velocity would be less than in the pure sliding case. In fact, its speed at the bottom of this incline would be $\sqrt{(10/7)\,gH}$ in the case of rolling.

Acknowledgments

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LOCAL GUIDE

The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 36." Do **not** ask for them by book title.

PROBLEM SUPPLEMENT

1. Perform the calculation referred to at the end of Sect. 2 to determine the torque and force needed to increase the length of the day by 1 hr. when this torque and force are applied for 24 hr.

2. In AF[9] answer Questions 6 and 7 (p. 228). Work Problems 11.7(a), 11.9 (assume all of the wheel's mass is at radius 0.60 m), 11.11, 11.17 (part a), 11.21, and 11.22.

   Answers:

   11.7(a): 14.9 rpm

   11.22: $a = 7.62\,\mathrm{m/s}$ (not $7.4\,\mathrm{m/s}$).

3. Referring to Fig. 1 in this module's *text*, find the force that must be exerted by the pivot on the bat at the instant the bat is struck at A. [Answer: Eq.(4) of the text.]

4. If the force F remains constant and perpendicular to the bat, find the resultant force on the bat at the pivot when the bat's angular speed is ω. [Answer: the resultant of F_p above and the centripetal force.]

5. Verify that the speed with which a sphere, rolling without slipping, leaves an incline (starting at rest from a vertical height H) is $\sqrt{(10/7)gH}$.

6. In AF work Problems 11.13, 11.17 (b and c), and 11.24.

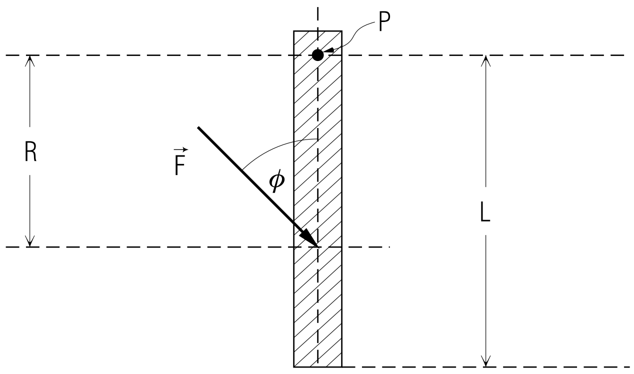
   Answers:

   11.24: (a) 120.05 J; (b) Tension in part of string attached to m is 35.2, to part attached to m' is 32.3 N.

[9] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970), Sect. 11.4, pp. 217-220. See this module's *Local Guide* for access to this reference.

Detailed Problems

1.



On a horizontal frictionless surface, a uniform rod of mass M and length L is pinned and held fixed at one end at point P, free to rotate in a horizontal plane about point P. A force, $\vec{F}$, constant in magnitude, is exerted on the rod such that the force maintains a constant angle, ϕ, with the rod. The force acts at distance R from P (see sketch). Under the action of this force the rod, starting from rest, begins to rotate.

a. Does it rotate with constant angular velocity? [K]

b. Why, or why not? [T]

c. What is the moment of inertia of the rod with respect to the axis through p (you may look this up in any text)? [F]

d. Relative to an axis through point P, perpendicular to the plane of the paper, find the external torque acting on the rod. [N]

e. Find the angular acceleration of the rod. [C]

f. Is this angular acceleration constant? [L]

g. If the rod started at rest at $t = 0$, find how much time it takes to rotate through one complete revolution. [R]

h. What is its angular speed at time $t = T$, where T is the time for one complete revolution in (g) above? [V]

i. What is its kinetic energy at time $t = T$. [O]

j. Use the Work-Energy Relation to find how much work was done by the external force F during the interval from $t = O$ to $t = T$. [H]

k. Evaluate $\vec{F} \cdot d\vec{r}$ to find the work done by $\vec{F}$ during the interval from $t = O$ to $t = T$ and compare with answer to (j). [B]

l. Draw a free-body diagram showing all of the forces acting on the rod at some instant during its motion. Don't neglect the force the pin exerts on the rod. Resolve these forces into components perpendicular to the rod and those tangential. [W]

m. At the instant that the rod has made one complete revolution what is the tangential acceleration of the center of mass of the rod? [E]

n. At that time T what is the centripetal acceleration of the center of mass? [J]

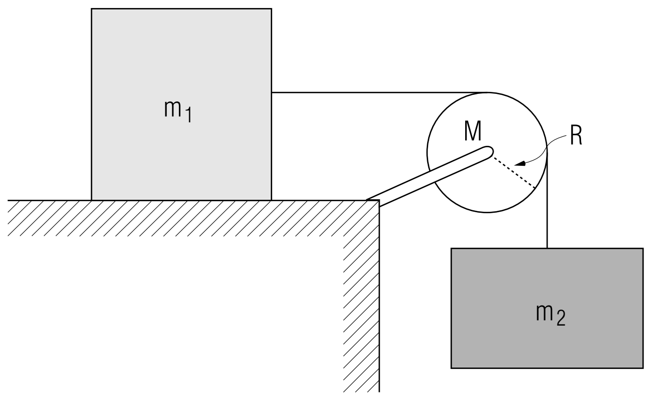
o. Write down that component of the expression of Newton's 2nd law that will enable you to determine the perpendicular component of the force the pin exerts on the rod. [U]

p. Determine this component of the force. [P]

q. Write that component of the expression of Newton's 2nd law that will enable you to determine the parallel to the rod component of the force the pin exerts on the rod. [A]

r. Determine this component of the force. [I]

2.



In the diagram above, the rope moves over the pulley wheel without slipping, turning the wheel. The frictional force between m_1 and the surface is $f = 12\,\text{N}$. The wheel has radius $R = 0.20\,\text{m}$, radius of gyration $K = 0.15\,\text{m}$ and mass $M = 20\,\text{kg}$. Also: $m_1 = 5\,\text{kg}$, $m_2 = 40\,\text{kg}$.

Find the acceleration of mass m_2, [M], the angular acceleration of the wheel, [D], the tension in the part of the cord attached to m_1, [Q], and the tension in that part of the cord attached to m_2 [G].

3. If the wheel of Problem 2 is on frictionless bearings so no frictional energy is lost in rotation of the wheel, use the Work-Energy Principle to find the speed of m_2 after it has fallen (starting from rest) a distance of $d = 2$ meters from its starting point. [S]

Brief Answers::

A. $F' - F\cos\phi = (ML/2)\cdot(12FR\sin\phi)/(ML^2)$.

B. Same as answer H.

C. $[(3FR)/(ML^2)]\sin\phi$.

D. $33.8/\sec^2$.

E. $[(3FR)/(2ML)]\sin\phi$.

F. $ML^2/3$.

G. 121.8 N.

H. It's equal to the change in kinetic energy: $2\pi FR\sin\phi$.

I. $F\cos\phi = (6\pi FR/L)\sin\phi$.

J. $[(6\pi)/(ML)]FR\sin\phi$.

K. No.

L. Yes.

M. $a = \dfrac{m_2 g - f}{m_1 + m_2 + \dfrac{MK^2}{R^2}} = 6.76\,\mathrm{m/s^2}$.

N. $FR\sin\phi$.

O. $2\pi FR\sin\phi$.

P. $F\sin\phi\left(1 - \dfrac{3R\sin\phi}{2L}\right)$.

Q. 45.8 N.

R. $\sqrt{(4\pi/3)\cdot(ML^2)/(RF\sin\phi)}$.

S. $v = \left[\dfrac{2d(f + m_2 g)}{m_1 + m_2 + \dfrac{MK^2}{R^2}}\right]^{1/2} = 5.36\,\mathrm{m/s}$.

T. There is a net external torque, causing an angular acceleration.

U. $F\sin\phi - F = M\left(\dfrac{3FR}{2ML}\sin\phi\right)$.

V. $\left[\dfrac{12FR\,\sin\phi}{ML^2}\right]^{1/2}$.

W.