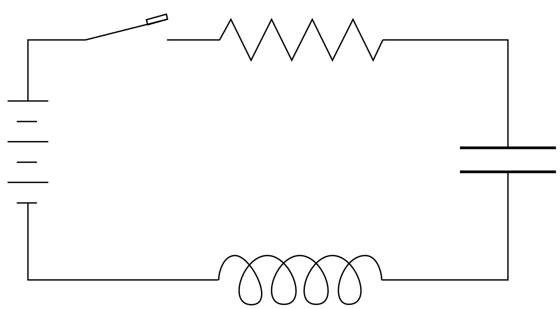
MISN-0-151

TWO-ELEMENT DC-DRIVEN SERIES LRC CIRCUITS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TWO-ELEMENT DC-DRIVEN SERIES LRC CIRCUITS

by
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Input Skills:

1. Describe the motion of a damped harmonic oscillator (MISN-0-29).
2. State the node and junction circuit rules (MISN-0-119).
3. State the relationship between voltage and charge in a capacitor (MISN-0-135).
4. State the relationship between voltage and current in an inductor (MISN-0-144).

Output Skills (Knowledge):

K1. Start with the relations for the potential drops across each of the three types of passive circuit elements and derive the relation between the current and the important circuit parameters for any two-element series DC-driven LRC circuit.

K2. List the mechanical analogs of the circuit components and important circuit parameters for two-element DC-driven LRC circuits.

K3. Given any two-element DC-driven LRC circuit, use analogies with the damped harmonic oscillator to sketch a graph of the time dependence of the charge on the capacitor or the current in the circuit.

Post-Options:

1. "Three-Element D.C.-Driven Series LRC Circuit" (MISN-0-152).
2. "Resonances in LRC Circuits" (MISN-0-154).

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New authors, reviewers and field testers are welcome.

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1. Introduction

Suppose we know how to analyze a circuit containing just capacitors, or just resistors, or just self-inductances. What happens when two of these three components are connected in series? In the present unit we derive the basic equations governing such circuits and describe the use of mechanical analogs in solving them. In another unit we treat the case where all three types of circuit elements are present, working toward the general analysis and synthesis of circuits.

2. Relating $Q(t)$ and Its Derivatives to L, R, C

The circuit shown in Fig. 1 is a closed loop. Thus:

$$V_0 - V_R - V_L = V_C, \qquad (1)$$

where V_R stands for the potential drop across the resistor, and similarly for the capacitor and inductor. We can apply Ohm's Law to the resistor and the corresponding relations for V_C and V_L.[1] Substituting these into Eq. (1) gives us:

$$V_0 = IR + L\dot{I} + \frac{Q}{C}, \qquad (2)$$

where dots stand for derivatives with respect to time. The signs show that all three elements are assumed to oppose the applied voltage V_0, meaning

[1]See MISN-0-135 and MISN-0-144.

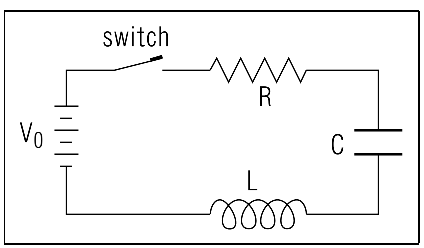


Figure 1. Switch is open for $t < 0$ Switch is closed for $t > 0$.

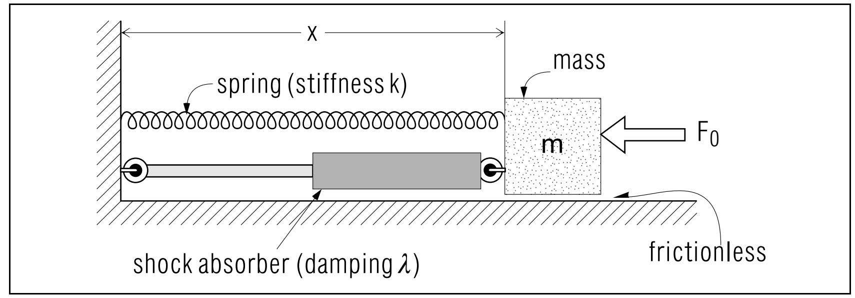


Figure 2. Mechanical Analog for the series LRC Circuit.

that the voltage source must do work to put charge on the capacitor, to put current through the resistor, and to change the current going through the inductor.

Using the definition of current, $I = dQ/dt$, we convert Eq. (2) to a relation for the charge on the capacitor:

$$V_0 = L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q. \qquad (3)$$

3. Mechanical Analogs

3a. The Mechanical Series Circuit. Compare Eq. (3) to the force equation for a damped driven spring[2] (see Fig. 2):

$$F_0 = m\ddot{x} + \lambda\dot{x} + kx\,, \qquad (4)$$

where k is the spring constant (a measure of its stiffness), m is its mass, and λ is the circuit's damping coefficient (this can include damping in the spring itself or may be mainly due to a damping device such as a shock absorber or a dashpot).

3b. Mechanical-Electrical Correspondences. Notice the similarity between Eqs. (3) and (4). The correspondences are listed in Table 1.

[2]See "The Damped Driven Oscillator," MISN-0-29.

Table 1: Analogies Between the *LRC* Circuit and the Damped Driven Oscillator.

LRC Circuit		Oscillator	
Inductor	L	Mass	m
Resistor	R	Shock Absorber	λ
Capacitor	C^{-1}	Spring	k
Applied Voltage	V_0	Applied Force	F_0
Charge	Q	Position	x
Current	$\dot{Q} = I$	Velocity	$\dot{x} = v$

Notice in Table 1 that a large capacitor is the analog of a weak spring. That is because, for a given amount of charge stored in it, a larger capacitor has less potential difference across it than does a smaller capacitor. Put another way, it is easier to stretch a *weak* spring to a larger displacement just as it is easier to put more charge into a *large* capacitance.

4. The *LC* Case

4a. The *LC* Circuit. We begin the examination of two-element circuits with the *LC* case, illustrated in Fig. 3. This case is especially interesting because it is like the perfect spring attached to a mass. That means there is no friction, no energy dissipation. In the electric circuit case it means no resistance so no energy dissipation. In either system, mechanical or electrical, an oscillation once started will go on forever. Of course in real life such systems are not possible, but one can come pretty close.

4b. Starting an Oscillation. In the mechanical case one starts an oscillation be applying a force to the mass so as to stretch the spring away from zero displacement from equilibrium. One then lets go of the mass, perhaps by opening one's fingers, and oscillations begin. In the electrical case one usually isolates the capacitor and charges it from a battery. One then disconnects the battery and connects the capacitor to the other inductor and the oscillations begin.

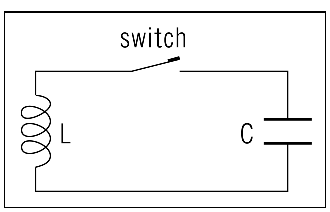


Figure 3. A two-element *LC* circuit

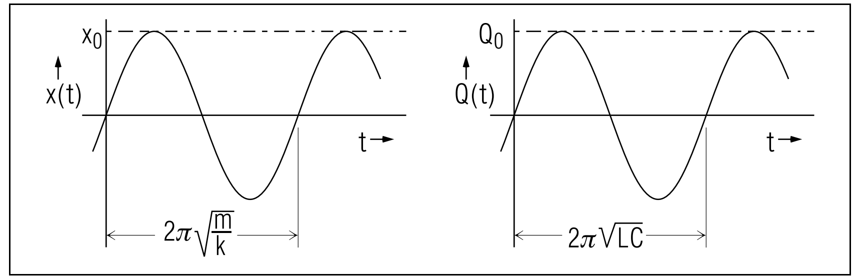


Figure 4. Comparison of solutions in the illustrative example.

4c. Oscillations. In the mechanical oscillations case, the *displacement* is a sinusoidal function of time, alternately taking on positive and negative values with respect to the equilibrium position. In the electrical oscillations case the *charge* on the capacitor is a sinusoidal function of time, alternately taking on positive and negative values. In the mechanical oscillations case the energy alternates between kinetic energy "stored" in the motion of the mass and potential energy "stored" in the compression or extension of the spring. In the electrical oscillations case the energy alternates between being stored in the inductor's magnetic field and in the capacitor's electric field.

4d. Mathematical Solution by Analogy. Consider the circuit shown in Fig. 3. Suppose the capacitor is charged by an external source, which is then removed at time $t = 0$. What happens to the capacitor charge as time progresses (switch closed)?

The equivalent mechanical model is just that of Fig. 2 after removal of the shock-absorber and the applied force ($R = 0$, $V_0 = 0$). This leaves a mass with a spring attached, where the mass is displaced from equilibrium and then released. The result is simple harmonic motion:

$$x(t) = x_0 \sin \omega t, \text{ where } \omega = \sqrt{k/m}.$$

Using our analogies in Table 1,

$$Q(t) = Q_0 \sin \omega t, \text{ where } \omega = \sqrt{1/LC}.$$

These two equations are graphed in Fig. 4.

The period of oscillation for the LC circuit is $P = 2\pi/\omega = 2\pi\sqrt{LC}$.

5. The *RL* and *RC* Cases

With three available elements, L, R, and C, there are three possible two-element circuits: LC, RL, and RC. The LC case was treated in Sect. 4; the other two are covered in this module's *Problem Supplement*.

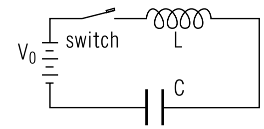
Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

1. The circuit for this problem is shown in the sketch below.

   a.



   Draw the mechanical analog for the circuit, omitting those parts of the general model shown in Fig. 2 that do not have corresponding elements in the circuit.

   b. In the mechanical case, the applied force does not affect the period or the amplitude of the oscillation of the mass; it merely shifts the equilibrium point about which the oscillation occurs. The time dependence of x is thus identical to the equation found in the example, except for an added constant and a phase shift:

$$x(t) = x_1 + x_0 \sin(\omega t + \phi)\,,$$

   where $x_1 = F_0/k$, and x_0 and ϕ depend on initial conditions. Using the appropriate analogies, write the equation for the time dependence of Q in the above LC circuit.

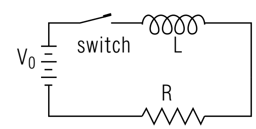
   c. Given $q(0) = 0$ and $I(0) = 0$ for the above circuit, can you evaluate Q_0 and/or ϕ in your answer to part (b) above? If so, what are they?

   d. Graph $Q(t)$ vs. t for this circuit.

   e. By taking the derivative of the charge function in both the illustrative example and your answer to part (b), find the shift in current caused by the introduction of the induced voltage in the above circuit. Assume $Q(0) = 0$ in both cases.

2. The circuit for this problem is shown in the sketch below.

   a.



   Draw the mechanical analog for the circuit, omitting those parts of the general model shown in Fig. 2 that do not have corresponding elements in the circuit.

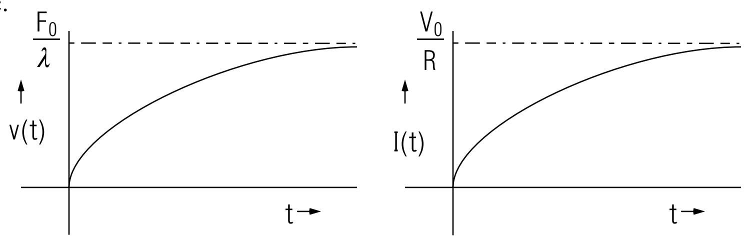
b. The shock absorber in the mechanical case resists any acceleration by exerting a force proportional to its rate of contraction. The proportionality constant is λ. Thus when the force is applied to the mass, it accelerates until the resistance force just equals the applied force. The compression then continues at a constant rate. In the electrical case, the induced voltage across the inductor causes an increase in current, which causes an increase in the potential difference across the resistor. As this difference approaches V_0, the net current change per unit time goes to zero, and the current approaches a constant value.

The equation for the velocity of the mass in the mechanical case is:

$$v(t) = \frac{F_0}{\lambda}\left(1 - e^{-\lambda t/m}\right) .$$

Write the corresponding equation for the electrical case.

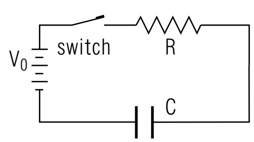
c.



Here are graphs of the two equations of part (b). Notice that velocity and current time are graphed, since there is no accumulating charge. What analogy can be made to the position of the mass in the mechanical model?

3. The circuit for this problem is shown in the sketch below.

a.



Draw the mechanical analog for the circuit, omitting those parts of the general model shown in Fig. 2 that do not have corresponding elements in the circuit.

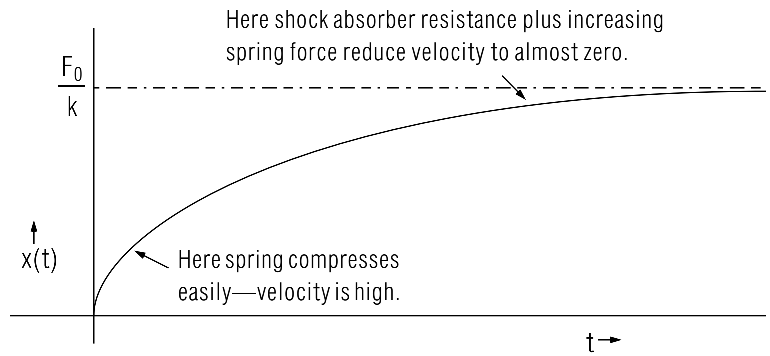
b. The force equation for the mechanical case is:

$$F_0 = \lambda\dot{x} + kx.$$

There is no mass, so the system will just shift until it is in static equilibrium. The shock absorber and spring work together to prevent the "overshoot" effect discussed in the illustrative example. This effect can be seen more easily by studying a graph of the solution, which is:

$$x(t) = \frac{F_0}{k}\left(1 - e^{-kt/\lambda}\right) .$$

Graphed, this is:



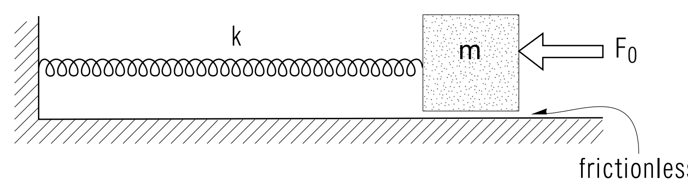
Note that on the left side of the curve the spring compresses easily and the velocity is high. On the right side of the curve the shock absorber resistance plus the increasing spring force reduce the velocity almost to zero.

▷ Now, using our analogy, write the time dependence of charge for the above circuit.

c. Sketch a graph of $Q(t)$ vs. t for this circuit.

Brief Answers:

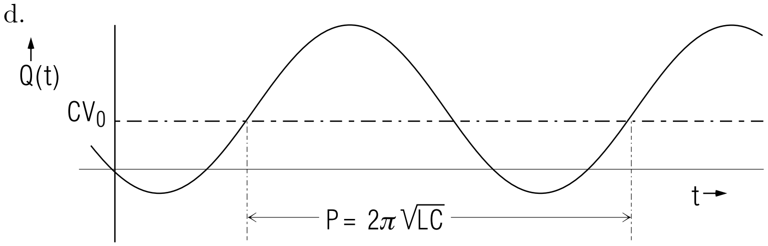
1. a.



b. $Q(t) = Q_1 + Q_0 \sin(\omega t + \phi)$, where $Q_1 = V_0 C$ and $\omega = \sqrt{1/LC}$.

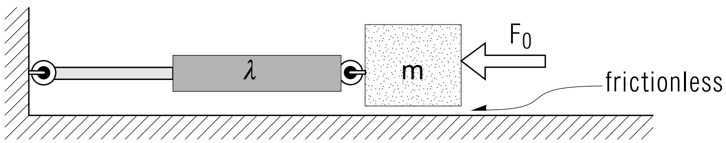
c. $\phi = \pi/2$. Q_0 must be found using a second initial condition.

d.



e. Zero.

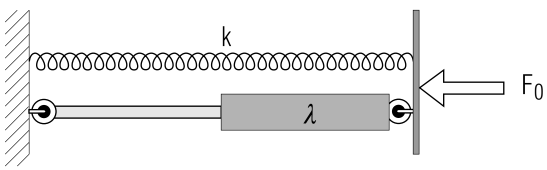
2. a.



b. $I(t) = (V_0/R)\ \left(1 - e^{-Rt/L}\right)$.

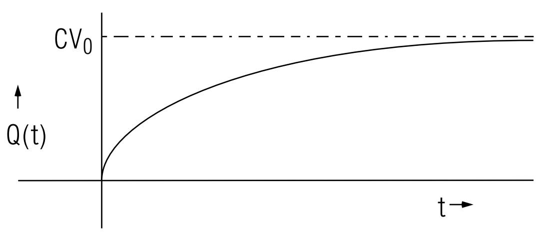
c. $I = \dot{Q}$ (the definition of current). So if we integrate I, at any point in the circuit, from 0 to t, we will get the net movement of charge in the circuit, Q.

3. a.



b. $Q(t) = CV_0\ \left(1 - e^{-t/RC}\right)$.

c.



MODEL EXAM

1. see Output Skills K1-K3 in this module's *ID Sheet*.

Brief Answers:

1. See this module's *text*.