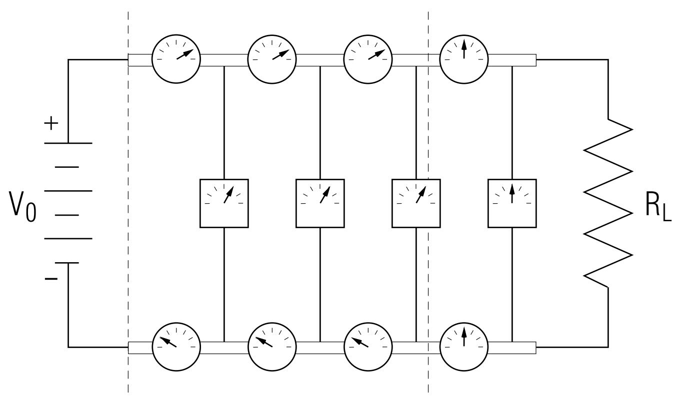
MISN-0-150

SIGNAL VELOCITY IN A CONDUCTOR



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SIGNAL VELOCITY IN A CONDUCTOR

by

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Title: **Signal Velocity in a Conductor**

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Input Skills:

1. State Ohm's Law (MISN-0-118).
2. State the two node rules for D.C. circuits (MISN-0-119).
3. Explain the conduction mechanism in wires (MISN-0-118).
4. Calculate the energy stored in the electric field of a capacitor (MISN-0-137).
5. Calculate the capacitance per unit length of a coaxial cable (MISN-0-135).
6. Calculate the inductance per unit length of a coaxial cable (MISN-0-144).
7. Calculate the energy stored in the magnetic field of an inductor (MISN-0-144).

Output Skills (Knowledge):

K1. When a battery becomes connected to one end of a coaxial cable, describe the subsequent traveling current and voltage waves for the cases: (a) proper termination at the far "load" end; (b) proper termination only at the "input" end; and (c) proper termination at neither end. Sketch figures that illustrate the waves.

K2. Outline the derivation of expressions for the velocity of transmission of a switch-closing signal along a transmission line and for the line's effective initial resistance, including the case of a coaxial cable (equations need only be named and/or described). State realistic numerical values for the two output quantities.

Output Skills (Rule Application):

R1. Given the characteristics of a battery, a coaxial cable, and a proper-termination load resistor, plus equations for the general case and for the capacitance and inductance per unit length along a coaxial cable, determine the values of the voltage and current waves that travel down the cable after the battery is connected to it and determine the speed of the waves.

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1. Introduction

It has been shown experimentally that the drift velocity of the electrons in a typical current in copper wire is a few feet per hour, yet the signal transmission appears to be virtually instantaneous. That is, when you talk to someone in San Francisco on the telephone, an electron which is activated in the voice coil of your telephone will not arrive in the receiver coil in San Francisco until about the year 2450! Nevertheless, you notice no such time lag between the end of your words and the beginning of the other person's reply. How can this be, since it takes moving electrons to power the receiver? In this module we explore the physics of a signal traveling down a two-wire circuit (a "transmission line") as an approximate model for signal-transmitting circuits in general.

2. Description

2a. Current and Voltage After a Switch Closes. When an electric field is introduced into a circuit by closing a switch, the mobile charges in all parts of the circuit do not immediately experience the force due to the electric field. Consider the simple circuit of Figure 1, where the switch connecting the battery to the transmission line has recently been closed. In the time since the switch was closed, the battery's voltage (its potential difference) has progressed a certain distance along the wires, and a current wave has accompanied it. We will derive the speed at which the voltage and current values move down the transmission line.

2b. Voltage-Current Relationship. As charges in successive parts of the transmission line are influenced by the electric field that accompanies the potential difference, those charges begin to drift into the adjacent region which had been neutral. The introduction of charge drift into new parts of the transmission line is represented by the "current front" that moves in step with the "voltage front." Note that no signal has yet reached the resistance R_L and, until it does, R_L has no influence over the voltage and current values proceeding toward it.

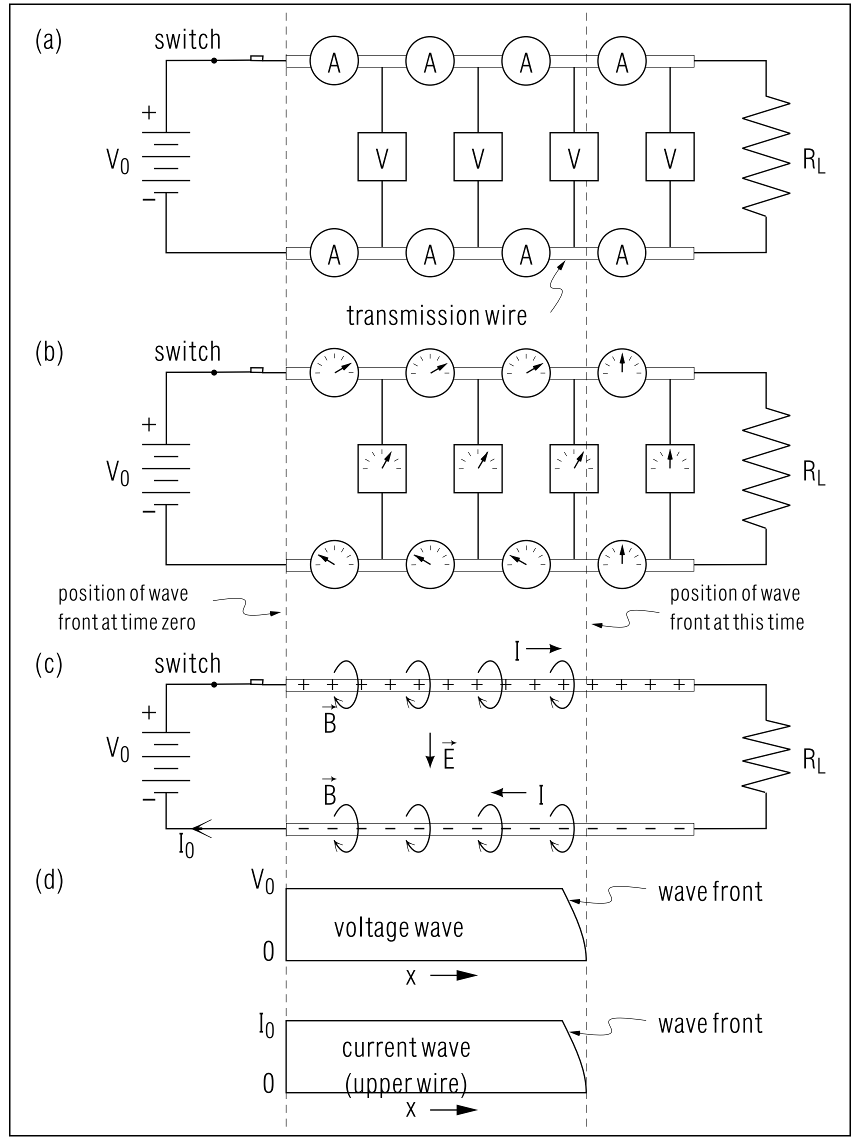

Figure 1. A simple transmission line (parallel wires) with the relevant parts of the wires greatly exaggerated in size. The voltmeters and ammeters are shown with their covers: (a) closed; and (b) open at a particular instant of time. For the same instant of time, the charges and electric and magnetic fields are shown in (c), the voltage and current waves in (d).

The current and voltage fronts constitute a signal activated by the closing of the switch. How are those fronts related and how fast do they move?

3. Current-Voltage Relation and Speed

3a. Energy Stored in Circuit. Let v_w be the speed at which the wave fronts in Fig. 1 advance. Then in time dt the fronts advance

$$dx = v_w \, dt\,,$$

and in that distance dx, an energy dE must be stored in the transmission line capacitance and inductance:

$$dE = \frac{1}{2}V^2\,dC + \frac{1}{2}I^2\,dL.$$

Here dC and dL are the capacitance and inductance in length dx of the transmission line, as shown in Fig. 2. If we call $\mathcal{C}$ the capacitance per unit length and $\mathcal{L}$ the inductance per unit length then $dC = \mathcal{C}\,dx$ and $dL = \mathcal{L}\,dx$ so we get for the increment of energy:

$$\begin{aligned} dE &= \frac{1}{2}V^2\mathcal{C}\,dx + \frac{1}{2}I^2\mathcal{L}\,dx \\ &= \frac{1}{2}\left(V^2\mathcal{C} + I^2\mathcal{L}\right)\,dx. \end{aligned}$$

Note that the inductance energy is stored in the form of a magnetic field, shown as circular arrows around the wires in Fig. 1. The capacitance energy is stored in the form of an electric field between the wires' excess charges.

3b. Energy Produced by Battery. The power produced by a battery in a D.C. circuit is the battery's voltage times the current it is sending out: $P = VI$ where $P = dE/dt$. Then the energy produced in time interval dt is:

$$dE = VI\,dt.$$

3c. Derivation of Propagation Velocity. Equating the energy produced by the battery to the energy stored in the circuit gives us the velocity of propagation of the front in terms of the voltage and current:

$$v_w = \frac{2VI}{V^2\mathcal{C} + I^2\mathcal{L}} \tag{1}$$

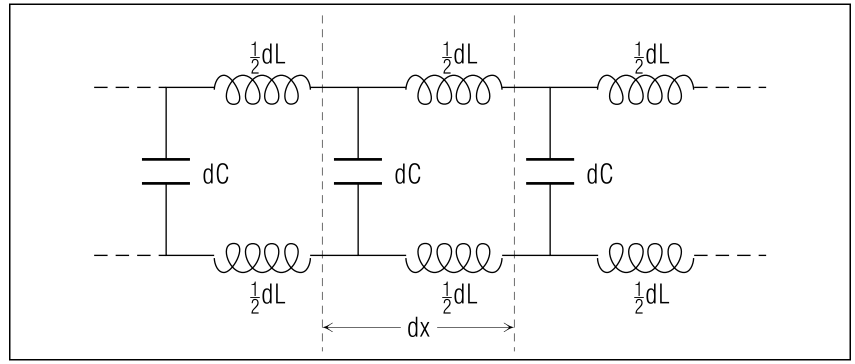

Figure 2. An isolated element dx of the circuit in Fig. 1, with resistance ignored.

The current can be eliminated by relating it to the charge sent out by the battery in time dt:

$$dQ = I dt,$$

which charges the capacitance in dx:

$$dQ = V dC = V\mathcal{C}\,dx.$$

Hence:

$$I = \mathcal{C}Vv_w, \tag{2}$$

where v_w is the wave front velocity. Using this to eliminate I in Eq. (1), we get the wave front velocity in terms of the transmission line characteristics:

$$v_w = (\mathcal{L}\mathcal{C})^{-1/2}. \tag{3}$$

3d. Example: Inductance and Capacitance. Now it remains to find typical values for self-inductance and capacitance along a typical transmission line. Commonly such lines are in the form of coaxial cables, as illustrated in Fig. 3. The self inductance per unit length of this type of conductor is[1]

$$\mathcal{L} = 2K_m k_m \ell n\,(b/a)\,, \tag{4}$$

where K_m is just slightly greater than unity for the insulation found in most coaxial cables. The capacitance per unit length for a coaxial cable

[1]See "Magnetic Inductance" MISN-0-144, for the derivation of this formula.

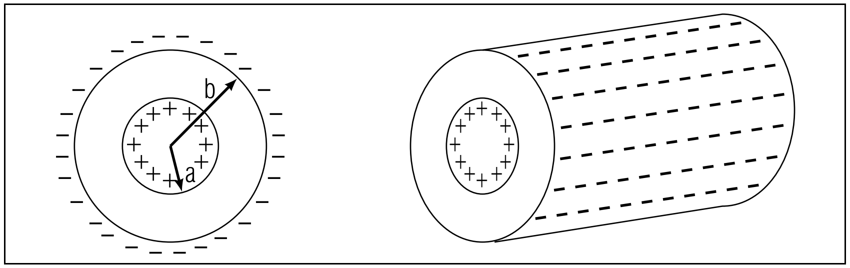

Figure 3. A coaxial cable, consisting of flexible concentric cylindrical conductors separated by flexible insulating material.

is[2]

$$\mathcal{C} = \frac{K_e}{2k_e \ell n\,(b/a)}\,, \tag{5}$$

where b is the radius of the outside conductor, a the radius of the inner one.

3e. Example: Velocity of Propagation. Combining Equations (4) and (5), we can get the wave front velocity in terms of common properties of a coaxial cable's dielectric insulator:

$$v_w = \frac{1}{\sqrt{\mathcal{C}\mathcal{L}}} = \frac{1}{\sqrt{(k_m/k_e)K_e K_m}} = \frac{c}{\sqrt{K_e K_m}}\,. \tag{6}$$

The symbols used above are: K_e is the dielectric constant of the insulating medium (1 for vacuum), K_m is the relative magnetic permeability of the insulating medium (1 for vacuum), k_e is the electric force constant, k_m is the magnetic force constant, and c is the speed of light. We have used the fact that the speed of light in vacuum, c, is related to the electric and magnetic force constants by:[3]

$$c = \sqrt{\frac{k_e}{k_m}}\,.$$

Since K_m and K_e are each just slightly greater than unity, for commonly used dielectrics, v_w is then slightly less than the speed of light c.

Note that v_w is independent of the coaxial cable's length and two radii!

[2]See "Electrostatic Capacitance" MISN-0-135, for a further discussion of the construction of coaxial cables and a derivation of their capacitance per unit length.

[3]See "The Derivation of the Electromagnetic Wave Equation from Maxwell's Equations" (MISN-0-210).

4. Time Development: Traveling Waves

4a. Overview. As described earlier and as illustrated in Fig. 1, traveling voltage and current waves go down the transmission line after the switch is closed. These waves reflect off the load and, altered by the load, head back up the line toward the source. When they reach the source they are reflected again and, altered by the source, head back down the line. The waves generally decrease with each reflection cycle and eventually a steady state is reached where the waves are gone and there is only a steady direct (DC) current determined solely by the source and load.[4]

4b. Deducing the Reflections. The characteristics of the reflection at either end of our transmission line can be deduced by applying Ohm's law to: (1) the incoming traveling voltage and current waves as they approach the end in question; (2) the outgoing traveling voltage and current waves after they leave the end; (3) the voltage and current values across the end before the waves reach it, again at the instant that both the incoming and outgoing traveling waves are at the end, and yet again after the outgoing waves have left the end. Note that after a traveling wave passes a point, the total voltage and current at that point are the values which were at that point before the wave arrived plus the traveling wave values. Thus a positive traveling wave adds to the values at points it passes (see Fig. 4). However, a positive current wave may subtract from the prior value at a point if that value represented a current in the opposite direction.

4c. Traveling-Wave Effective Resistance. We define the traveling-wave effective resistance of the line, denoted R_{eff}^{trav}, as the ratio of the value of the traveling voltage to the traveling current. From Eqs. (2) and (6), that value is:

$$R_{eff}^{trav} \equiv \frac{V}{I} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = 59.96\,\Omega\,\sqrt{\frac{K_m}{K_e}}\,\ell n\,(b/a)\,. \qquad \textit{Help: [S-1]}\,,$$

where Ω is the symbol for *ohms*. Since the symbolic factors are not greatly different from unity, the effective resistance of such a line for traveling waves will not be greatly different from 60 ohms. This traveling wave effective resistance is not to be confused with the line's direct current resistance, which we have assumed to be zero.

[4]Ideal inductors have zero resistance and ideal capacitors have infinite resistance in steady DC circuits so it is as though they were no longer in the circuit. Also, we have assumed that there is zero resistance in the line itself.

▷ Let us assume a coaxial cable connected to a 60 volt battery that has an internal resistance of 60 ohms. Assume the 60 ohm value for R_{eff}^{trav}. Show that the initial current wave traveling down the line will leave behind it 0.5 ampere through the line (going one direction in one conductor, the other direction in the other conductor), and the initial voltage travelling down the line will leave behind it 30 volts between the two conductors ("across the line"). *Help: [S-2]*

4d. Reflection at the End. When a traveling wave reaches the end of the transmission line, the existence of the load there, R_L in Fig. 1, and the necessity of Coulomb's Law to be obeyed there, causes a different-valued wave to head back up the line toward the source end. We will label the traveling voltage wave reaching the end $V_{end,in}^{trav}$ and the traveling voltage wave leaving the end $V_{end,out}^{trav}$. Similarly for the traveling current waves, $I_{end,in}^{trav}$ and $I_{end,out}^{trav}$. The resistance at the end will be labeled R_{end}. Then the incoming and outgoing waves are related by:

$$V_{end,out}^{trav} = \frac{R_{end} - R_{eff}^{trav}}{R_{end} + R_{eff}^{trav}} \, V_{end,in}^{trav} \, .$$

The ratio of the two waves is called the *reflection coefficient* and is written Γ_{end}:

$$\Gamma_{end} = \frac{R_{end} - R_{eff}^{trav}}{R_{end} + R_{eff}^{trav}} \, .$$

Then

$$V_{end,out}^{trav} = \Gamma_{end} \, V_{end,in}^{trav}$$

and

$$I_{end,out}^{trav} = \Gamma_{end} \, I_{end,in}^{trav} \, .$$

These equations hold for both ends of the transmission line, even though there is an $\mathcal{EMF}$ at one end and not at the other.

▷ Show that for an open-ended line, where nothing connects the two conductors at the load end, the reflected wave travels back up the line doubling the voltage and canceling the current produced by the initial wave. *Help: [S-3]*

▷ Show that for a shorted-end line, where the two wires are connected by zero resistance, the reflected wave travels back up the line canceling the voltage and doubling the current produced by the initial wave.

▷ Show that for a *properly terminated* line, where the two wires are connected by a resistance equal to the traveling wave effective resistance, there is no reflected wave. *Help: [S-4]*

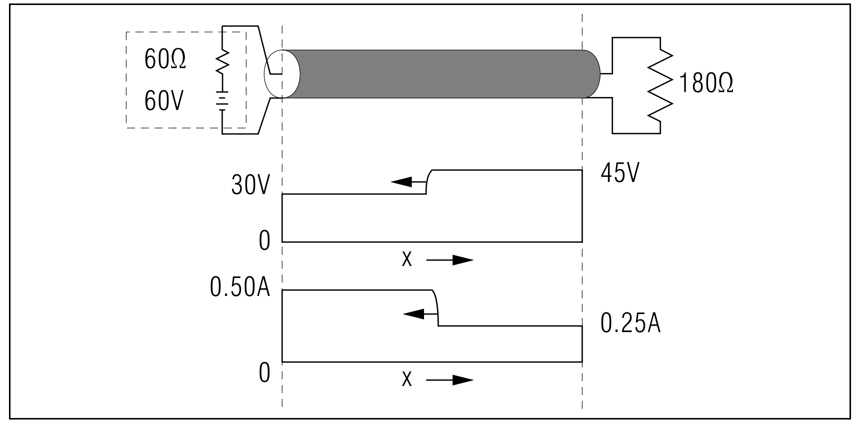

Figure 4. Reflected voltage and current waves for a case where the terminating resistor is larger than the line's effective resistance (assumed to be 60 ohms).

▷ Show that when the source resistance is equal to the traveling wave effective resistance, there is no reflection at the source end.

4e. Proper Termination. Reflected waves are considered harmful in transmission lines because reflections from earlier signals add to, and hence interfere with, later signals. Designers try to choose a proper value for the terminating resistance so that such reflections do not occur. However, if the resistances at the source and terminus are *improper*, reflections and reflections of reflections will continue to travel up and down the line. Figure 4 shows an example of improper termination at the load end but proper termination at the source end. Note that the initial wave down the line is a (60 volt, 0.50 ampere) wave, while the reflected wave is a (15 volt, 0.25 ampere) wave. The reflected wave leaves in its wake 45 volts and 0.25 ampere. If the line has proper termination at the source end, there will be no reflection when the wave reaches it so the voltage and current values along the cable will then be steady.

5. Conclusion and Caveat

5a. Checking Against Observation. The transmission speed, close to the speed of light, checks with the observed (or unobserved!) transmission speed of your voice from coast to coast. In fact, the delay time due to the finite transmission speed is so short that you are usually unable to perceive it. Many cross-country telephone transmissions now occur through ground-to-satellite-to-ground transmission: such transmission is mainly through thin air so the speed of transmission is indeed close to c. However, the signal may still go from your receiver to the upload transmitter by ordinary wires.

5b. Resistive Losses Neglected. Note that in the above derivation we have neglected the small slowing effects of resistance, and of periodic signal-boosting circuitry that is needed to overcome such attenuation of the signal. We have also ignored the fact that loads and sources generally have inductance and capacitance. Finally, we have not considered real communication signals going down the transmission line. All of these effects are dealt with in real-life designs.[5]

Acknowledgments

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[5] See, for example, *Principles of Radar*, By Members of the Staff of the Radar School, Massachussetts Institute of Technology, Third Edition by J.F. Reintjes and G.T. Coate, McGraw-Hill Book Co., Inc., New York (1952).

PROBLEM SUPPLEMENT

Note: Problem 1 also occurs in this module's *Model Exam*.

1. A coaxial cable has an inner conductor of radius 0.500 mm and an outer one of radius 5.00 mm. The insulating material between the conductors is polyethylene so its dielectric constant, K_e, is 2.30 (Table 1, MISN-0-135) and its relative magnetic susceptibility, K_m, is 1.00 because polyethylene is non-magnetic. It is fed by a 1.20×10^2 V battery that has an internal resistance of 0.500 Ω. At the other end of the cable there is a $1.00 \times 10^3\,\Omega$ resistance.

   a. Determine the inductance and capacitance per unit length along the cable.

   b. Determine the speed of transmission of the signal wave along the cable.

   c. Determine the effective initial resistance of the cable.

   d. Determine the heights of the voltage and current waves that go down the cable from battery to resistance.

   e. Determine the heights of the voltage and current waves that return from the resistance toward the battery.

   f. Determine the final (steady-state) voltage and current along the cable.

2. A transmission line has a source with 80 volts and an internal resistance of 80 ohms. The effective resistance of the line for travelling waves is 80 ohms and the terminating resistance is 160 ohms.

   a. Determine the travelling voltage and current that first travels down the line.

   b. Determine the reflection coefficient at the terminus.

   c. Determine the travelling voltage and current that travels back toward toe source upon reflection of the first wave.

   d. Determine the voltage and current on the line after the reflected wave has passed.

   e. Determine the reflection coefficient at the source.

f. Check that the answers to part (c) satisfy Ohm's law for the all times beyond when the reflected wave reaches the source.

Brief Answers:

1. a. $\mathcal{L} = 4.6052 \times 10^{-7}\,\text{H/m} \Rightarrow 0.461\,\mu\text{H/m}$

   $\mathcal{C} = 5.5570 \times 10^{-11}\,\text{F/m} \Rightarrow 55.6\,\text{pF/m}$

   b. $0.659\,c$

   c. $91.04\,\Omega \approx 91.0\,\Omega$

   d. 119 V, 1.31 A *Help: [S-2]*

   e. 99.1 V, −1.09 A

   f. 218 V, 0.22 A

2. a. 40 V, 2.0 A

   b. 0.33

   c. 13.3 V, -0.67 A

   d. 53.3 V, 1.33 A

   e. zero

   f. (check it)

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-4c)*

Using values on Appendix page A6-1 of *Introductory Physics for Engineers and Scientists, Vol. 2* (Haydn-McNeil Publishing, Plymouth, MI, 1997):

$$2\sqrt{k_e k_m} = 59.96\,\frac{\text{N m}}{\text{s A}^2} = 59.96\,\Omega$$

and note that $\ell n\,(b/a)$ will be 1 if b/a is 2.73.

S-2 *(from TX-4c, PS-problem 1d)*

This is just a case of two resistors dividing a voltage produced by the internal source in a battery (one of the two resistors is the internal resistance of the battery).

S-3 *(from TX-4d)*

A gap has an infinite resistance in vacuum and almost so in air. As the resistance is raised yoward infinity, the ratio of the numerator to denominator in Γ approaches unity (1.00).

MODEL EXAM

$$k_e = c^2 k_m$$

$$\frac{V}{I} = \sqrt{\frac{L}{C}}; \qquad v_w = \frac{1}{\sqrt{\mathcal{C}\mathcal{L}}}$$

$$\text{coax:} \qquad \mathcal{L} = 2K_m k_m \ell n\,(b/a); \qquad \mathcal{C} = \frac{K_e}{2k_e \ell n\,(b/a)}$$

1. See Output Skills K1-K2 in this module's *ID Sheet*. The actual exam may contain either or both of these skills, or neither.

2. A coaxial cable has an inner conductor of radius 0.500 mm and an outer one of radius 5.00 mm. The insulating material between the conductors is polyethylene so its dielectric constant, K_e, is 2.30 (Table 1, MISN-0-135) and its relative magnetic susceptibility, K_m, is 1.00 because polyethylene is non-magnetic. It is fed by a 1.20×10^2 V battery with an internal resistance of 0.50 Ω. At the other end of the cable there is a $1.00 \times 10^3\,\Omega$ resistance.

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   d. Determine the heights of the voltage and current waves that return from the resistance toward the battery.

   e. Determine the final (steady-state) voltage and current along the cable.

Brief Answers:

1. See this module's *text*.

2. See problem 1 in this module's *Problem Supplement*.