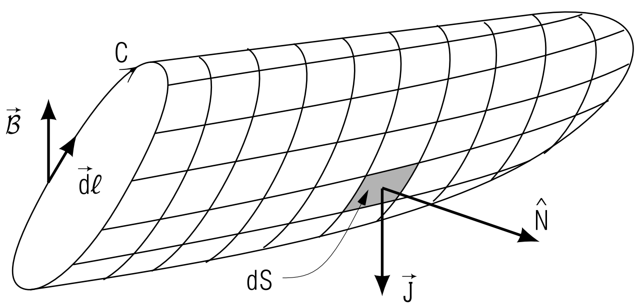
MISN-0-145

THE AMPERE-MAXWELL EQUATION; DISPLACEMENT CURRENT



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THE AMPERE-MAXWELL EQUATION;
DISPLACEMENT CURRENT

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Title: **The Ampere-Maxwell Equation; Displacement Current**

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Input Skills:

1. Skills from "Ampere's Law" (MISN-0-138).

Output Skills (Knowledge):

K1. Demonstrate how Ampere's Law is modified when the electric field vector varies with time.

Output Skills (Problem Solving):

S1. Apply the Ampere-Maxwell equation to situations where even if there is no actual current in a region of space, the changing electric field induces a magnetic field whose circulation is given by the Ampere-Maxwell equation.

External Resources (Required):

1. M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970); for access see this module's *Local Guide*.

Post-Options:

1. "The Faraday-Henry Law of Magnetic Induction" (MISN-0-142) deals with a phenomenon that is complementary to the one dealt with here. The subject is the electric fields that arise when magnetic fields vary with time.
2. "Maxwell's Equations" (MISN-0-146).

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1. Description

Ampere's Law is an integral theorem that relates the line-integral of the magnetic field $\vec{B}$ around a closed path to the total electric current enclosed by that path. This theorem is not complete if there are electric fields in the vicinity which vary with time. The modification of Ampere's Law including the effect of the changing electric field is the subject of this module.

2. Suggested Procedure

In AF[1] study Sections 20.9 and 20.10.

Note that $\epsilon_0 \equiv 1/(4\pi k_e)$ and $\mu_0 \equiv 4\pi k_m$.

Work problems 20.20 and 20.21 in AF.

Work Problems *A* and *B* in the next section.

Read pages 832-840 of K. W. Ford's "Classical and Modern Physics," Vol. 2. For access, see this module's *Local Guide*. Examples 1 and 2 on pages 833-835 are especially enlightening.

3. Exercises

A. The plates of a parallel-plate capacitor are circular disks of radius 0.040 meters, as shown in Fig. 1.

The plates, initially at a potential difference of 1.00×10^3 V are discharged through a 10.0 ohm resistor.

a. Just before the discharge begins, when the capacitor potential is constant, what is the magnetic field at a point 0.020 meters from the central axis of the capacitor (between the plates)?

[1] M. Alonso and E. J. Finn, *Physics*, Addison-Wesley, Reading (1970). For access, see this module's *Local Guide*.

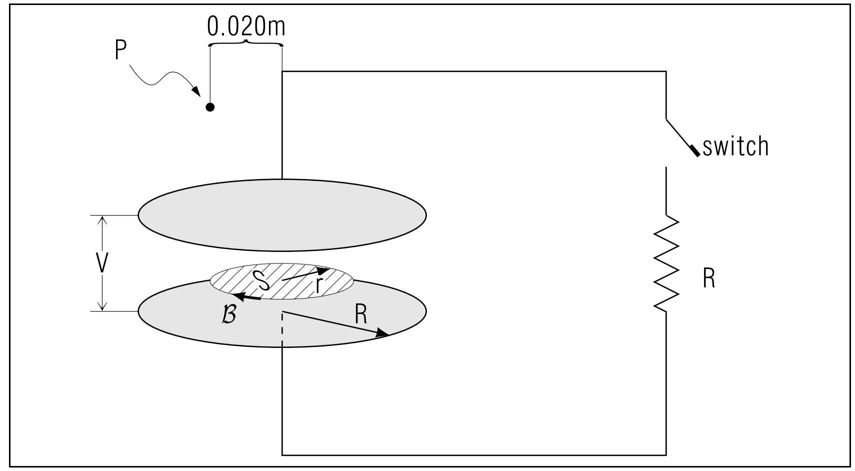


Figure 1. The capacitor of Sect. 3, Exercise A.

b. At this same instant what is it 0.020 meters from the central axis of the capacitor but outside of the region between the plates? (At point *P* in the figure above. Assume the wires connecting the capacitor to the resistor join the plates at the center of the circular disks, the wires coinciding with the central axis of the capacitor).

c. At the instant the discharge begins (the instant after the switch is closed) a current begins to flow, reducing the charge on the capacitor. At that instant, what is the magnetic field at a point 0.020 meters from the central axis, between the plates of the capacitor? [Refer to MISN-0-151 for the time dependence of a current in a circuit containing only a capacitance and a resistance. Also see Fig. 21.15 and equation below equation 21.18 on page 487 of AF. What you want is the current very near $t = 0$. Also you need to use Gauss's Law (MISN-0-133), to get the electric field between the capacitor plates].

d. At that same instant as in (c), find the magnetic field at point *P*, 0.020 meters from the axis of the capacitor (the wire) but outside of the plates. Compare this with the field at the corresponding point within the capacitor [the answer to part (c)].

B. Show that the product $\mu_0\epsilon_0$ both numerically and dimensionally equals $(1/c^2)$ where c is the speed of light.

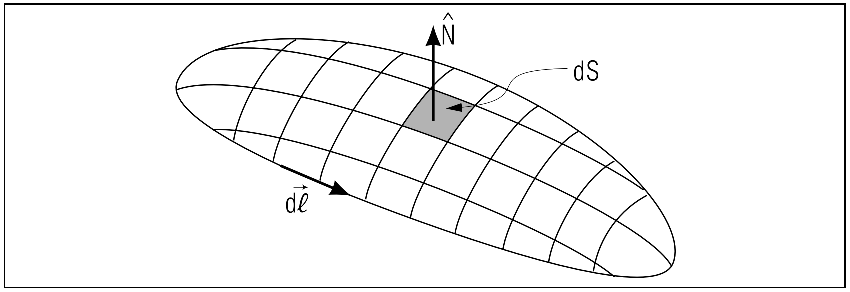


Figure 2. An illustration of an element of surface area.

4. Comments

Refer also to the COMMENTS section of MISN-0-138.

Ampere's Law for distributed currents is written as:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot \hat{N}\, dS,$$

where $\vec{j}$ is the value of the current density (amperes per square meter whose direction is the direction in which charge is transported at the point where $\vec{j}$ is evaluated), $\vec{j} \cdot \hat{N}$ is the component of $\vec{j}$ perpendicular to the element of area dS (see Fig. 2). The circle on the integral means that the integral is around a *closed* path. The fact that there is not a circle on the surface integral means that it is not over an entire *closed* surface (more on this later).

Now observe, in Fig. 3, that in the steady state the current crossing the shaded area A is the same as the current crossing the curved area—any curved area which has the path C in common with the plane area A. Thus in Ampere's Law,

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot \hat{N}\, dS\,,$$

any surface defined by curve C can be used for the surface integral on the right (see Fig. 4). (Note that the direction of the normal, $\hat{N}$, and the path of integration C are related by the right-hand rule. The directions of $\vec{B}$ and $\vec{j}$, of course, are determined by the physics of the situation and remain the same even when $d\vec{\ell}$ and $\hat{N}$ are both reversed).

For a closed surface S there is no curve C, as illustrated in figure 20.19, and if there is a steady-state current entering the surface at one

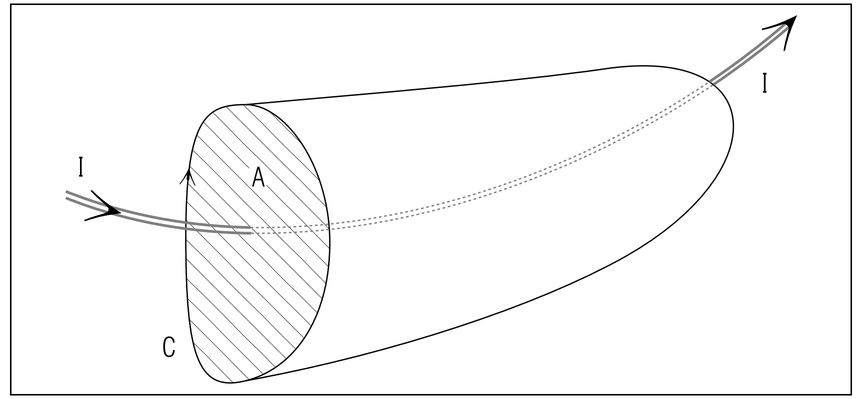


Figure 3. An illustration of the application of Ampere's Law.

point and leaving it at another, the surface integral of $\vec{j}$ over the closed surface is zero:

$$\oint_S \vec{j} \cdot \hat{N}\, dS = 0.$$

This is just the statement of the conservation of charge, if there is no gain or loss of charge inside the volume enclosed by the surface. [See also section 20.9 of the *Readings*].

Now consider, as an example, a system consisting of two parallel metallic plates separated by a distance D with positive charge on one and negative charge on the other (see Fig. 5).

The wires coming out of the negative plate and out of the positive plate are not connected (a gap between x and y) so the charges stay fixed

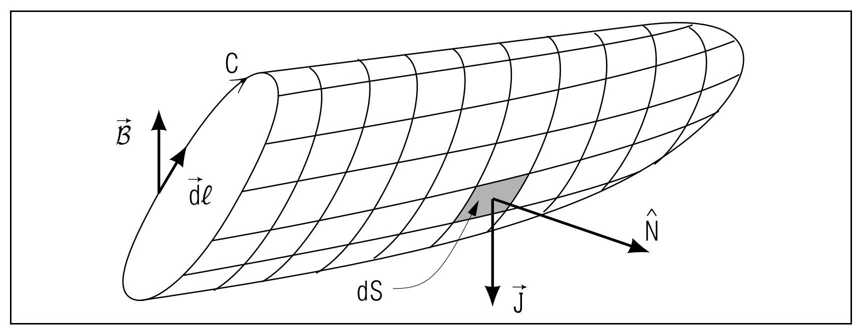


Figure 4. Physical elements of Ampere's Law.

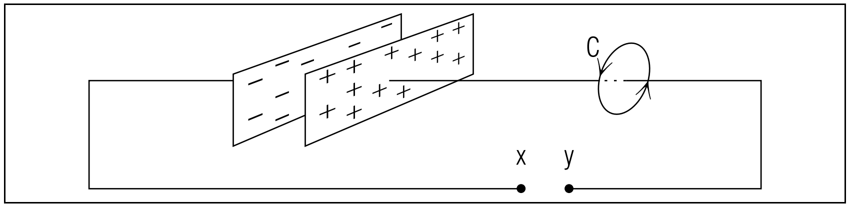


Figure 5. A system of two charged, seperated, parallel plates.

on the plates and no current flows in the wires. Thus if this system is isolated there is no magnetic field anywhere and the line integral of $\vec{B}$ around curve C (enclosing the wire) is zero:

$$\oint_C \vec{B} \cdot d\vec{\ell} = 0\,,$$

because no current crosses any surface enclosed by path C.

Even if we deform the plane surface enclosed by C to include the positive plate, the above result is, of course, still true (see Fig. 6).

Now if points x and y are brought together, the negative charge will begin to flow counter-clockwise (the current will flow clockwise from the positive to the negative plate). So there is a current crossing the *plane surface* S enclosed by C. Ampere's Law says (see Fig. 7):

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \vec{j} \cdot \hat{N}\, dS\,.$$

However, there is no charge crossing the curved surface S', so if we used this surface S' to evaluate the right side we would get zero. If we used the plane surface S, also enclosed by C, the integral gives the

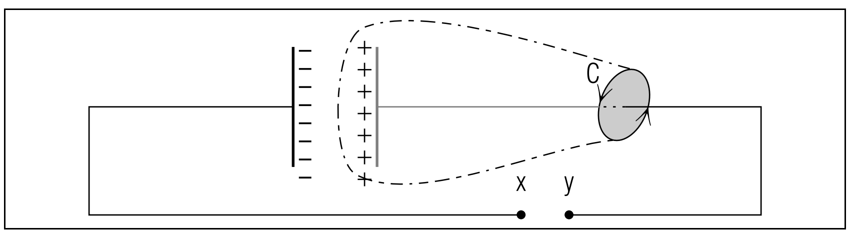


Figure 6. The system of Fig. 5 showing a chosen closed path C.

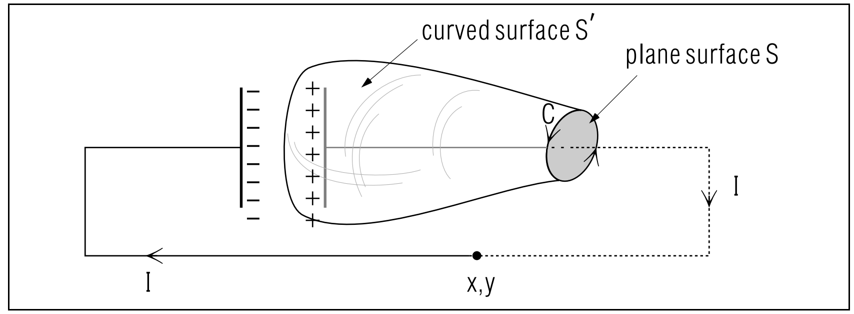


Figure 7. The system of Fig. 5 with the circuit now completed.

instantaneous value of the current crossing S. So there appears to be an inconsistency.

To resolve this inconsistency, there must be something else that the line integral of $\vec{B}$ depends upon besides the current density $\vec{j}$. The difference between this situation and the steady current case is that in this case the current is not a steady current. The current decreases as the charge on the plates decreases and goes to zero when the plates are no longer charged. The electric field, $\vec{E}$, between the plates also changes with time and it is this changing electric field crossing the curved surface S' which must be included on the right side of Ampere's Law to satisfy the law for non-steady state conditions.

To see this we need to relate the surface integral of the electric field vector $\vec{E}$ over a closed surface to the charge enclosed by that surface. This relation you recall, is embodied in Gauss's Law:

$$\oint_S \vec{E} \cdot \hat{N}\, dS = \frac{Q_S}{\epsilon_0},$$

where Q_S is the charge *inside* the volume enclosed by the closed surface S.

If the charge inside the enclosed volume changes with time, it must flow across the surface (this follows from charge conservation):

INCREASE in charge (per second) inside volume	=	current flowing across surface *into* volume	−	current flowing across surface *out of* volume

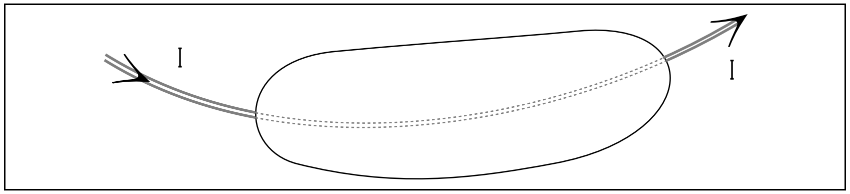


Figure 8. A section of a steady current.

$$\frac{dQ_S}{dt} = \oint_S \vec{j}_{in} \cdot \hat{N}\, dS - \oint_S \vec{j}_{out} \cdot \hat{N}\, dS.$$

where we have separated the current density into $\vec{j}_{in}$ and $\vec{j}_{out}$). If we have a steady state, then:

$$\frac{dQ_S}{dt} = 0 = \oint_S [\vec{j}_{in} + \vec{j}_{out}] \cdot \hat{N}\, dS = \oint_S \vec{j} \cdot \hat{N}\, dS = 0,$$

as before.

For our circuit, Q_S is decreasing, there is only current flowing out of the volume (across the area enclosed by loop C) so:

$$\frac{dQ_S}{dt} = -\oint_S \vec{j}_{in} \cdot \hat{N}\, dS.$$

From Gauss's Law:

$$\epsilon_0 \oint_S \vec{E} \cdot \hat{N}\, dS = Q_S,$$

where the integral on the left is over the *closed* surface enclosing the positive plate (both the curved and the plane surface are defined by curve C).

Taking the derivative of both sides of this with respect to t:

$$\epsilon_0 \frac{d}{dt} \oint_S \vec{E} \cdot \hat{N}\, dS = \frac{dQ_S}{dt}.$$

So we have:

$$\epsilon_0 \frac{d}{dt} \oint_S \vec{E} \cdot \hat{N}\, dS = -\oint_S \vec{j}_{out} \cdot \hat{N}\, dS,$$

where the integral on both sides is over the same closed surface. Combining both terms on the left side:

$$\oint \left[\epsilon_0 \frac{d\vec{E}}{dt} + \vec{j}_{out}\right] \cdot \hat{N}\, dS = 0, \qquad (1)$$

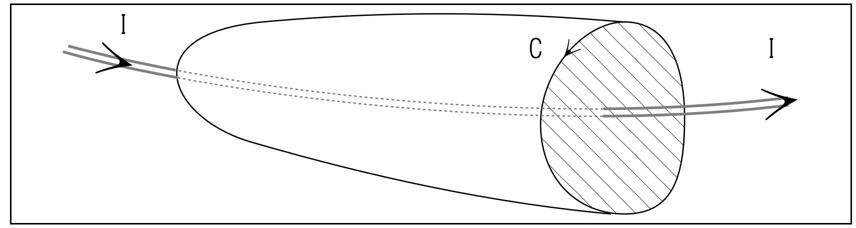


Figure 9. Ampere's Law employed in the case of Fig. 8.

where $\hat{N}$ is the *outward* normal at every element dS over the closed surface.

This is true if what we have is *not* a steady current but a situation where net charge is leaving the enclosed region. Recall that for a steady current (where no net charge accumulates or leaves the region enclosed; see Fig. 8):

$$\oint_S [\vec{j}_{in} + \vec{j}_{out}] \cdot \hat{N}\, dS = 0 = \oint_S \vec{j} \cdot \vec{N}\, dS.$$

Comparing these last relations, we can see that $\epsilon_0(d\vec{E}/dt)$ piays the same role that a current density, $\vec{j}$, does.

So recapping, here is Ampere's Law for the case where we have steady currents (see Fig. 9):

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot \hat{N}\, dS,$$

where the surface S is either the plane or curved surface bounded by the curve C.

Now refer to Fig. 10 where the charge is leaving the region enclosed by the curved and plane surfaces.

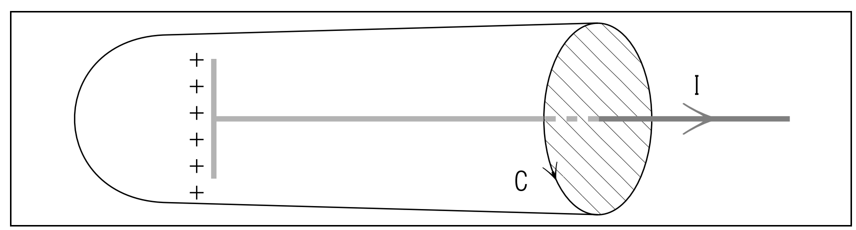


Figure 10. The case where charge is leaving the enclosed region.

Applying Eq. (1):

$$\oint_S \left[\epsilon_0 \frac{d\vec{E}}{dt} + \vec{j}_{out}\right] \cdot \hat{N}\, dS = 0.$$

We break up this integral over the closed surface into surface integrals over the curved (cu) and planar (pl) parts:

$$\int_{pl} \vec{j}_{out} \cdot \hat{N}\, dS + \int_{cu} \vec{j}_{out} \cdot \hat{N}\, dS \int_{pl} \epsilon_0 \frac{d\vec{E}}{dt} \cdot \hat{N}\, dS + \int_{cu} \epsilon_0 \frac{d\vec{E}}{dt} \cdot \hat{N}\, dS = 0.$$

The second integral is zero because there is no current density crossing the curved surface. The third integral is zero because no E fields cross the planar (flat) surface. Hence,

$$\int_{pl} \vec{j}_{out} \cdot \hat{N}\, dS = -\epsilon_0 \int_{cu} \frac{d\vec{E}}{dt} \cdot \hat{N}\, dS.$$

where in each surface integral $\hat{N}$ is the normal pointing out of the region enclosed by the surfaces. If over the curved surface $\hat{N}$ points to the "right" or into the enclosed region, then the sign of the right side should be made positive. Hence, using Ampere's Law,

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot \hat{N}\, dS = \mu_0 \epsilon_0 \int \frac{d\vec{E}}{dt} \cdot \hat{N}\, dS. \quad (2)$$

where the second integral above is over the part of the plane surface that is enclosed by C (not a closed surface) and the third integral is over the curved surface *if* $\hat{N}$ points to the right, same as for the plane surface.

Eq. (2) says that, in the case where there is no steady current, the line integral of $\vec{B}$ around a closed curve is equal to the surface integral of $(\mu_0 \epsilon_0 d\vec{E}/dt)$ over the surface traced by C (where again the direction of the integral around C and the direction of $\hat{N}$ are related by the right-hand rule).

Eq. (2) is Ampere's Law for the situation where there is no steady current, but $\vec{E}$ varies with time. If both are present (steady current and changing $\vec{E}$ field), we have:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \int_S \frac{d\vec{E}}{dt} \cdot \hat{N}\, dS. \quad (3)$$

This is the Ampere-Maxwell relation, a companion to the Faraday-Henry Law:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{N}\, dS. \quad (4)$$

These two, with Gauss' Law for Electric and Magnetic fields,

$$\oint_S \vec{E} \cdot \hat{N}\, dS = \frac{Q_S}{\epsilon_0}, \quad (5)$$

where Q_S is the charge inside the closed surface S, and:

$$\oint_S \vec{B} \cdot \hat{N}\, dS = 0, \quad (6)$$

The four equations, Eqs. (3)-(6), make up the set called Maxwell's Equations. (MISN-0-146 enlarges upon this).

5. Brief Answers to Assigned Problems

20.20 The current out of the cube is $I = -12a^3\epsilon_0 t$ if one corner of the cube is at the origin.

20.21 The answer assumes the cube is located as shown in Fig. 11. The answer AF gives has $(1/c^2)$ instead of $\mu_0\epsilon_0$, where c is the speed of light. The two answers are the same.

Problem A

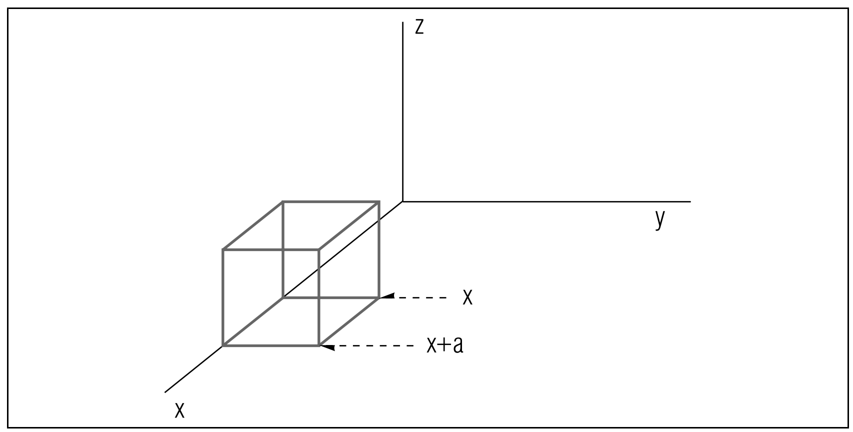


Figure 11. The physical situation of problem 20.21

a. zero

b. zero

c. In an RC circuit with the capacitor initially charged to potential V, the current $I = (V/R)e^{-t/RC}$, decreasing exponentially with time. At $t \approx 0$, when the switch is just closed, $I \approx V/R$. Between the plates,

$$\epsilon_0 E = \frac{Q}{\pi r_0^2}\,,$$

where Q is the charge on either plate and πr_0^2 is the area of the plate (see the parallel plate capacitor in MISN-0-133). Then $\epsilon_0(dE/dt) = 1/(\pi r_0^2)$.

Using the Ampere-Maxwell equation between the plates, we get for B, 0.02 meters from the axis: $|\vec{B}| = 2.5 \times 10^{-4}$ teslas.

d. $|\vec{B}| = 1.0 \times 10^{-3}$ teslas.

Problem B See text.

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

LOCAL GUIDE

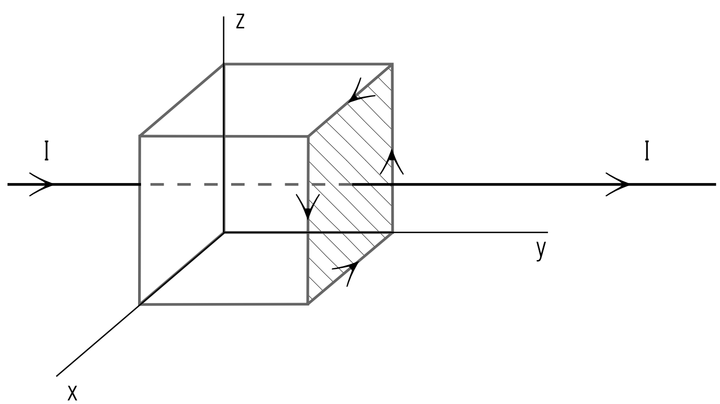
The readings for this unit are on reserve for you in the Physics-Astronomy Library, Room 230 in the Physics-Astronomy Building. Ask for them as "The readings for CBI Unit 145." Do **not** ask for them by book title.

PROBLEM SUPPLEMENT

Note: Problem 2 also appears in this module's *Model Exam.*

$$\epsilon_0 \equiv 1/(4\pi k_e) \qquad \text{and} \qquad \mu_0 \equiv 4\pi k_m$$

1.



Consider the cube above whose sides are 2 meters.

In this region the eletric field vector is given by:

$$\vec{E} = at[(5-y)\hat{y} - z\hat{z}],$$

where a is a constant, t is the time in seconds, y and z are the coordinates along the y- and z-axes. In addition, there is a thin wire carrying a steady current I along the y-direction, entering the cube at the left and leaving the cube at the surface on the right.

a. What is the total charge enclosed by the cube? [F]

b. Is it increasing or decreasing? [C]

c. What is the line integral of $\vec{B}$ around the shaded surface in the direction indicated? [H]

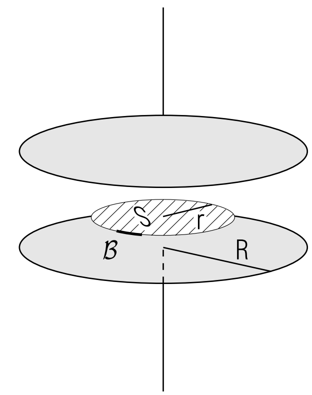
d. What is the line integral of $\vec{B}$ around the top surface (in the direction determined by the right hand with the thumb pointing in the positive z-direction)? [A]

e. How much net current crosses the shaded area? E]

f. What net current enters the left face of the cube? [B]

g. What net current crosses the front face of the cube? [I]

2.



Above is a capacitor formed of a pair of very large parallel plates. They are closely spaced so the parallel-plate capacitor formula can be used to compute the electric field $\vec{E}$ at all radii out to R. The capacitor is being charged so a current I flows onto the top plate and away from the bottom one.

Around the path enclosing the shaded circular area S, because of the symmetry of the situation, we can safely assume $\vec{B}$ is tangent to the circle of radius r and constant at all points on the circle.

Use the Ampere-Maxwell relation to find the Magnetic field at points inside this plate. [You'll need to use Gauss' Law to find $\vec{E}$ as a function of the current I] [D]

What is B when the capacitor is fully charged and the $\vec{E}$ inside is uniform? [G]

Brief Answers:

A. $-8\mu_0\epsilon_0 a$.

B. $I - 20\epsilon_0 a$.

C. It is decreasing linearly with time.

D. $B = \dfrac{\mu_0 I r}{2\pi R^2}$.

E. $I + 12\epsilon_0 a$.

F. $Q = -16\epsilon_0 a t$.

G. zero.

H. $-8\mu_0\epsilon_0 a$.

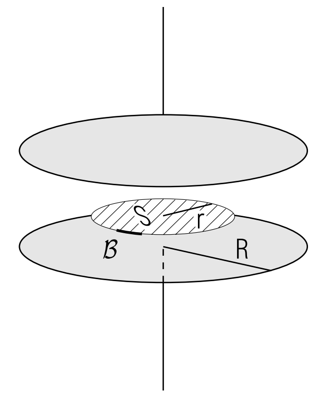
B. $\mu_0 I + 12\mu_0\epsilon_0 a$.

I. zero.

MODEL EXAM

1. See Output Skills K1 and S1 in this module's *ID Sheet*. The actual exam may have one or both of these skills, or neither.

2.



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Around the path enclosing the shaded circular area S, because of the symmetry of the situation, we can safely assume $\vec{B}$ is tangent to the circle of radius r and constant at all points on the circle.

Use the Ampere-Maxwell relation to find the Magnetic field at points inside this plate. [You'll need to use Gauss' Law to find $\vec{E}$ as a function of the current I]

What is B when the capacitor is fully charged and the $\vec{E}$ inside is uniform?

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 2.