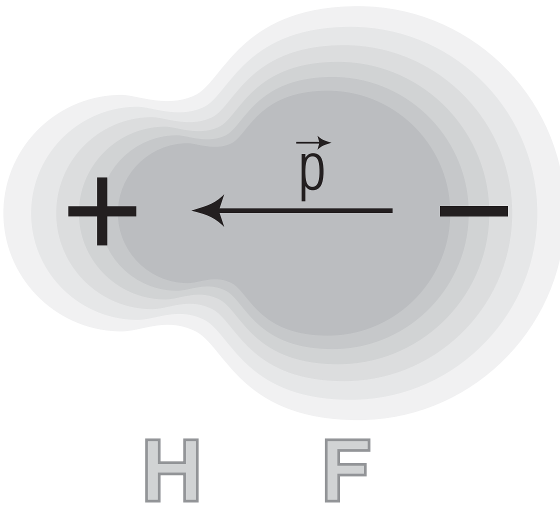
PHYSNET

MISN-0-120

ELECTRIC DIPOLES



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ELECTRIC DIPOLES

by

P. Sojka and J. S. Kovacs

Title: **Electric Dipoles**

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Input Skills:

1. State Coulomb's Law (MISN-0-114).
2. Describe the field of a point charge (MISN-0-115).
3. Describe the potential due to a point charge (MISN-0-116).
4. Define vector dot products and cross products (MISN-0-2).

Output Skills (Knowledge):

K1. Define the dipole moment of a discrete system of charges.

K2. Define a "point dipole."

Output Skills (Problem Solving):

S1. Given two point charges and a distance between them, calculate the dipole moment, electric field, and electric potential they produce.

S2. Calculate the electric field and electric potential due to a given point dipole.

S3. Calculate the potential energy of a given dipole at a given orientation in an external electric field.

S4. Calculate the torque on a given dipole in a given external electric field and the work done when the dipole is in the external electric field.

Post-Options:

1. "Magnetic Dipoles" (MISN-0-130).

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New authors, reviewers and field testers are welcome.

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1. Introduction

Dipoles play interesting and important roles in atomic and subatomic physics, as well as in Chemistry and Engineering. On the atomic and subatomic scale, the magnetic and electric dipole moments reveal structural information about the systems. On a much larger scale, dipole antennae are important in radio transmission and reception.

In this module, we will discuss one of the two types of dipoles, the electric dipole. Under consideration will be the dipole moment, the field and potential due to a dipole, torque on a dipole due to an external electric field, the electric potential and the potential energy of a dipole in an external electric field, and the work done on a dipole by the field when the dipole is rotated.

2. Properties of Electric Dipoles

2a. Definition of Dipole Moment. Consider a collection of N charges, q_1, q_2, …, q_N. Relative to a fixed coordinate system, each charge is located at a point described by a vector $\vec{r}_1$ for q_1, $\vec{r}_2$ for q_2, etc. (see Fig. 1). The dipole moment of this system is defined as the vector

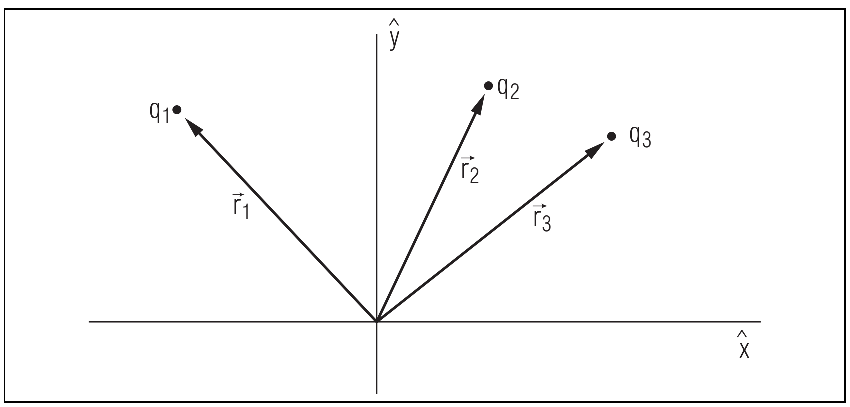

Figure 1. Position vectors for a system of charges.

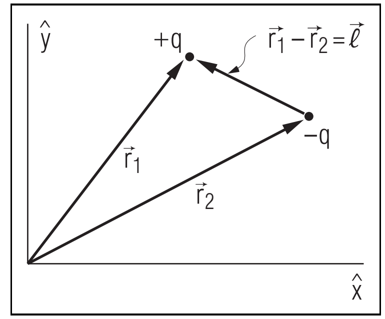

Figure 2. Vectors used to define an electric dipole.

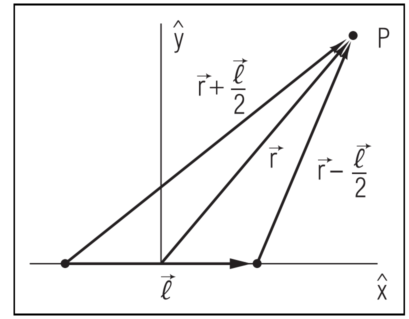

Figure 3. Position vectors for the electric field of a dipole.

sum of the position vectors weighted by the charges at the ends of the vectors:

$$\text{Dipole Moment Vector } \equiv q_1\vec{r}_1 + q_2\vec{r}_2 + \ldots + q_N\vec{r}_N \equiv \sum_{i=1}^{N} q_i\vec{r}_i. \qquad (1)$$

2b. Definition of A Dipole. "A dipole" is defined as a system consisting of two equal but oppositely charged point charges: $q_1 = q$ and $q_2 = -q$. For that case the dipole moment, $\vec{p}$, is:

$$\vec{p} = q_1\vec{r}_1 + q_2\vec{r}_2 = q\vec{r}_1 - q\vec{r}_2 = q(\vec{r}_1 - \vec{r}_2) = q\,\vec{\ell}, \qquad (2)$$

where $\vec{r}_1 - \vec{r}_2 = \vec{\ell}$, the vector separation of the charges, points from the negative charge $(-q)$ to the positive charge $(+q)$ (see Fig. 2). Note that the origin of the coordinate system, which appears in the general definition, disappears in the case of this two-particle, zero-net-charge dipole. Thus the dipole moment is a property of the dipole and is independent of what coordinate system is used to describe it.

2c. Electric Field Due to a Dipole. The electric field and the electric potential at any point in the vicinity of a dipole can be straightforwardly calculated just by adding the contributions due to each of the charges. For example, consider a dipole whose center is at the origin (Fig. 3). At point P you can calculate the field due to the two charges. The answer is:

$$\vec{E}(\vec{r}) = k_e q \left[\frac{\vec{r} - \vec{\ell}/2}{|\vec{r} - \vec{\ell}/2|^3} - \frac{\vec{r} + \vec{\ell}/2}{|\vec{r} + \vec{\ell}/2|^3} \right], \qquad (3)$$

where $\vec{r}$ is the vector from the origin to P.

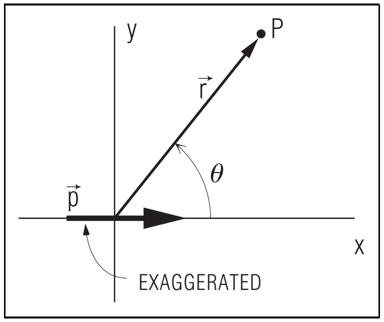

Figure 4. Symbols used to define the potential at a point P due to an electric dipole of moment $\vec{p}$ at the origin.

2d. Electric Field Due to a "Point" Dipole. An interesting result occurs if we take the dipole to be "very small." Here we mean that the dipole has a spatial size, ℓ, that is negligible compared with the distance r to the point P where the field of the dipole is observed. The result is that the field at P due to the point dipole at the origin is well-approximated by:[1]

$$\vec{E}(\vec{r}) = k_e \left[\frac{(3\vec{r}\cdot\vec{p})\,\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] \qquad (\ell \ll r)$$

Note that there is one component in the $\vec{r}$ direction and another in the $\vec{p}$ direction. The expression becomes the exact answer for $\vec{E}$ as the ratio ℓ/r becomes vanishingly small, regardless of whether $\vec{p}$ and $\vec{E}$ become small or not.

2e. Potential Due to a "Point" Dipole. The potential, $V(r)$, can be similarly determined, yielding (see Fig. 4 for symbols):[2]

$$V(r) = k_e \frac{\vec{p}\cdot\vec{r}}{r^3} = k_e \frac{p\cos\theta}{r^2}.$$

This is the potential at a distance r from a point-like dipole. The interesting thing about these expressions is that they depend neither on the charge nor the spatial size, but on the combination $\vec{p}$, the dipole moment. Notice also that the potential varies with θ. For example, at any point on a line perpendicular to the direction of $\vec{p}$ the potential V is zero. For a given value of r, V has its maximum value for the point where r is in the direction of p. Contrast that with the potential due to a point charge at the origin: for that case, V has the same value for a fixed $|\vec{r}|$, no matter what the direction of $\vec{r}$.

[1]See Appendix A for an outline of the derivation.
[2]See Appendix B for the derivation.

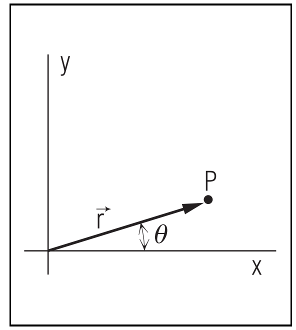

Figure 5. Coordinate-space symbol definitions for all points at constant radius.

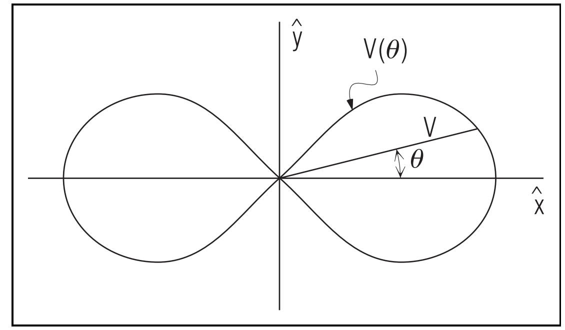

Figure 6. Dipole potential-space symbols for points at constant radius, varying angle.

2f. Polar Plot of $V(\theta)$. This dependence of V on θ can be graphically exhibited in a polar plot of V as a function of θ. For each value of θ a line is drawn whose length equals the magnitude of V at that value of θ. The dipole potential is:

$$V(\theta) = k_e \frac{P\cos\theta}{r^2},$$

where r is now held constant at some particular radius of interest.

For the point P in Fig. 5, the value of $V(\theta)$ is given by the length of the line from the origin (at angle θ) shown in Fig. 6. If p were oriented along the y-axis, the graph would be rotated 90°. On this same kind of plot, the potential due to a point charge ($V = k_e q/r$) is a circle. Another point of contrast between the potential due to a point charge and the potential due to a point dipole is that the former potential decreases as r^{-1}, while the latter decreases as r^{-2}.

2g. Summary: Field and Potential of a Dipole. In spite of the way we define it in terms of point charge, the point electric dipole can be viewed as another kind of electrical entity, to be contrasted with and considered along with a point charge, as a source of electric field. Even though it has a net charge of zero, it does give rise to an electric field and to a corresponding electric potential and both are basically different from the field due to a point charge. Just as

$$E_q = k_e \frac{q\hat{r}}{r^2} \qquad \text{and} \qquad V_q = k_e \frac{q}{r}$$

are the field and potential characteristic of a point charge,

$$\vec{E}(\vec{r}) = k_e \left[\frac{(3\vec{r}\cdot\vec{p})\,\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right] \qquad \text{and} \qquad V(\vec{r}) = k_e \frac{\vec{p}\cdot\vec{r}}{r^3}$$

are the field and potential characteristic of a point dipole.

3. Dipole In External Electric Field

3a. Potential Energy. Now suppose a point dipole is placed in an external electric field, $\vec{E}_{\rm ext}$. What happens to the dipole (does it move)? What is the new energy of the dipole? Let us consider a simple external field, a uniform field along the positive x-axis. Suppose the dipole has its center at the origin and makes an angle θ with the x-axis in the x-y plane. For the sake of easy calculations, we will deal with a finite-size dipole instead of the point dipole. The result we get will apply equally well to the point dipole.

The external electric field has associated with it a potential:

$$\vec{E}_{\rm ext} = -\vec{\nabla} V_{\rm ext}.$$

Because $\vec{E}_{\rm ext}$ is constant and in the x-direction,

$$\vec{E}_{\rm ext} = -dV_{\rm ext}/dx\,\hat{x}; \qquad \frac{dV}{dx} = -E; \qquad V_{\rm ext} = -(x - x_0)\,E_0,$$

where x_0 is some arbitrary integration constant. The coordinates of the charges q and $-q$ are (See Fig. 7):

$$\vec{r}_q = \frac{\ell}{2}(\hat{x}\cos\theta + \hat{y}\sin\theta),$$

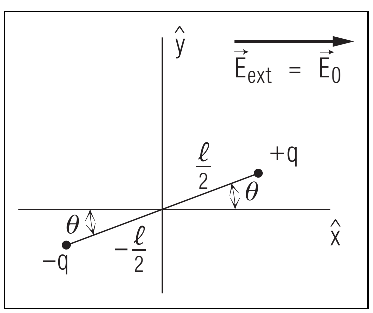

Figure 7. Definition of variables specifying a dipole in an external field.

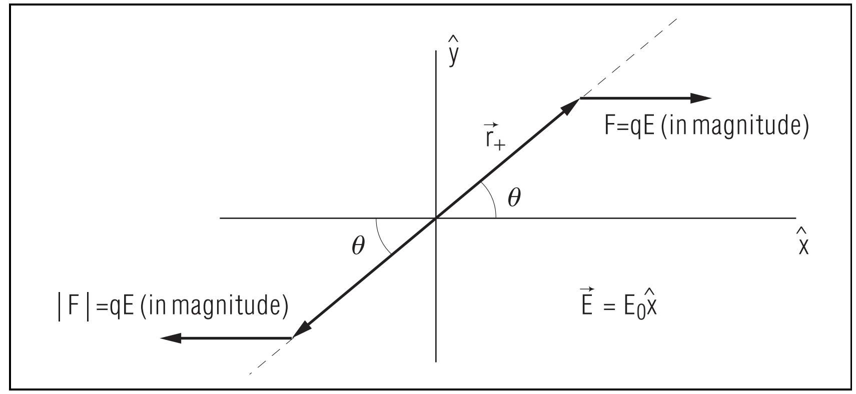

Figure 8. Symbols used for forces on a dipole in a uniform electric field.

$$\vec{r}_{-q} = \frac{\ell}{2}(\hat{x}\cos\theta - \hat{y}\sin\theta),$$

The two charges q and $-q$ in the external potential $V_{\rm ext} = -E_0(x - x_0)$ then have a total potential energy given by the sum of $qV_{\rm ext}$ values at the two charge positions:

$$E_{\rm pot} = -q\ell E_0\cos\theta = -pE_0\cos\theta = -\vec{p}\cdot\vec{E}_{\rm ext}.$$

This expression for the potential energy of a dipole in an external electric field, $E_{\rm pot} = -\vec{p}\cdot\vec{E}_{\rm ext}$, while derived for the special case of a uniform field, also applies to the general case (any field).

3b. Torque on a Dipole in an External Electric Field. Consider a dipole in a uniform electric field, oriented at an angle θ with the $\vec{E}$-direction. The forces on the dipole are as shown in Fig. 8. The torque about the center of the dipole, exerted on the dipole by the force on the $+q$ end, is $\vec{r}_{+q}\times\vec{F}$ (where $\vec{r}_{+q}$ is the vector from the origin, the dipole's center, to the $+q$ charge). Then $\vec{r}_{+q}$ is of length $\ell/2$ so the magnitude of the torque is $(1/2)q\ell E\sin\theta$. The direction of this torque is perpendicular to and into the plane of the page. The contribution to the net torque by the force on the other end of the dipole is also $(1/2)q\ell E\sin\theta$, directed into the page, so the net torque on the dipole is $q\ell E\sin\theta$, the result you'd get from the expression $\vec{p}\times\vec{E}$. The direction of the torque vector also checks with the direction of $\vec{p}\times\vec{E}$ so we can write:

$$\vec{\tau} = \vec{p}\times\vec{E}.$$

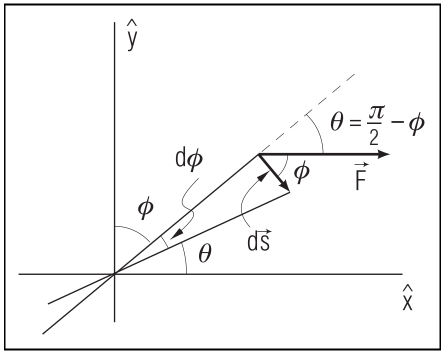

Figure 9. Quantities involved in the rotation of a dipole through a small angle $d\phi$.

There is no net force on this dipole ($\vec{F}$ to the right, and $-\vec{F}$ to the left) but there is a net torque, so that the effect of the external E-field will be to cause the dipole to rotate.

3c. Determination of Rotational Potential Energy. Consider rotation of the dipole about its center at the origin of the coordinate system. How much work will the E-field do in rotating the dipole from an orientation with angle θ_A to an orientation θ_B?

In general, if force $\vec{F}$ causes a system to be displaced by an amount $d\vec{s}$, the work done by $\vec{F}$ is $\vec{F} \cdot d\vec{s}$. For the charge $+q$ at the end of the dipole, the displacement $d\vec{s}$ (for rotation through $d\theta$ about the origin) is perpendicular to the dipole moment vector.

So $\vec{F} \cdot d\vec{s} = F ds \cos\phi$ (where ϕ is as shown in Fig. 9) but $ds = (\ell/2)\, d\phi$ and $F = qE$ so the increment of work done by $\vec{E}$ is $(qE\ell/2)\cos\phi\, d\phi$, which, from the expression for the torque, we recognize as $\tau\, d\phi$. The work done in rotating the dipole from orientation A making angle ϕ_A with the field to orientation B making angle ϕ_B with the field is:

$$W = \int_A^B \vec{F} \cdot d\vec{s} = \int_A^B \tau\, d\phi = E\, q\, \ell \int_A^B \cos\phi\, d\phi.$$

Note that the (1/2) is gone because the total torque on the dipole also includes the torque exerted on the $-q$ end. Then:

$$W = E\, p(\sin\phi_B - \sin\phi_A).$$

But $\phi_B = \pi/2 - \theta_B$, so: $\sin\phi_B = \cos\theta_B$, etc.

$$W = (\vec{E} \cdot \vec{p})_B - (\vec{E} \cdot \vec{p})_A; \qquad W = E_{\rm pot}(\theta_A) - E_{\rm pot}(\theta_B),$$

just the difference in potential energy between the initial orientation and the final one (recall that the potential energy of a dipole in an external field is: $E_p = -\vec{E} \cdot \vec{p}$). This is the work-energy principle in action.[3]

4. Existence of Electric Dipoles

Do electric dipoles really exist? Electric monopoles (electric "charges") certainly do. For most purposes an electron can be considered an example of a point monopole with charge $q = -1.6 \times 10^{-19}$ C. It also has mass and a magnetic dipole moment.[4] The proton also carries an electric monopole charge $q = +1.6 \times 10^{-19}$ C. Obviously, the hydrogen atom consisting of a proton and an electron forms an electric dipole. In its equilibrium state, the average location of the electron and proton coincide, yielding a dipole moment of zero. Atoms in the ground state normally have zero dipole moments. However, in some of their excited states, which last for brief time periods, many atoms have non-zero dipole moments. Elegant symmetry theorems in quantum mechanics explain why this must be so, but that's a subject for another time.

Of course, microscopic static charge distributions, such as might be present on capacitor plates, may have finite, non-zero, dipole moments.

Acknowledgments

Michael Cason and Orion Ciftja kindly led us to correct errors in the manuscript. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

A. The Point Dipole Field

(for those interested)

[3] See "Potential Energy; The Work-Energy Principle" (MISN-0-20).
[4] See "Magnetic Dipoles" (MISN-0-130).

Expand the denominators of Eq. (3) in a power series in (ℓ/r):[5]

$$\begin{aligned} |\vec{r} \pm \vec{\ell}/2|^{-3} &= (r^2 \pm \vec{r}\cdot\vec{\ell} + \ell^2/4)^{-3/2} \\ &= r^{-3}\left(1 \pm \frac{\vec{r}\cdot\vec{\ell}}{r^2} + \frac{\ell^2}{4r^2}\right)^{-3/2} = r^{-3}\left(1 \mp \frac{3\vec{r}\cdot\vec{\ell}}{2r^2} + \ldots\right). \end{aligned}$$

Discard all second and higher powers of the very small quantity (ℓ/r); the rest is algebra.

B. The Point Dipole Potential

(for those interested)

We define the zero of potential to be at infinity:

$$\begin{aligned} V(\vec{r}) &= -\int_\infty^{\vec{r}} E(\vec{r}\,')\cdot d\vec{s}\,' \\ &= -k_e \int_\infty^{\vec{r}} \left[\frac{(3\vec{r}\,'\cdot\vec{p})\,\vec{r}\,'}{r'^5} - \frac{\vec{p}}{r'^3}\right] d\vec{s}\,'. \end{aligned}$$

where $d\vec{s}'$ is a trajectory element. The potential is independent of the integration path. For convenience we choose the path to be along the radial line passing through the point $\vec{r}$. This line is at an angle θ to the dipole moment vector $\vec{p}$ of the dipole at the origin. Then:

$$\begin{aligned} V(r,\theta) &= -k_e \int_\infty^{r} \left[\frac{3\,r'\,p\,\cos\theta r'}{r'^5} - \frac{p\,\cos\theta}{r'^3}\right] dr' \\ &= -k_e 2\,p\,\cos\theta \int_\infty^{r} (r')^{-3}\,dr' \\ &= -k_e \frac{p\,\cos\theta}{r^2}. \end{aligned}$$

[5] See "The Taylor Series for the Expansion of a Function About a Point" (MISN-0-4).

PROBLEM SUPPLEMENT

Note: Problem 4 also occurs in this module's *Model Exam.*

1. Consider a dipole in an external electric field $\vec{E} = E_0\hat{y}$ where E_0 is a constant. For the four cases of orientation of the dipole (p is the magnitude of the dipole moment):

   (i) $\vec{p} = p\hat{x}$
   (ii) $\vec{p} = p\hat{y}$
   (iii) $\vec{p} = -p\hat{x}$
   (iv) $\vec{p} = -p\hat{y}$

   a. Find the potential energy of the dipole in this external field. Order your answers from highest to lowest potential energy.

   b. If the dipole were free to rotate in this external electric field, which orientation would be the equilibrium alignment? (as a hint, consider the analogy of a particle in the gravitational field of the earth; its potential energy increases with increasing altitude above the surface. If left free, in what direction of change of potential energy does it go?)

2. A point dipole $\vec{p} = p_0\hat{y}$ is located at the origin of the coordinates and $p_0 = 1.00 \times 10^{-15}\,\mathrm{C\,m}$. At a distance of 0.01 meters from the dipole, find the electric field at each of the following locations and, on a diagram, sketch in the direction of the field at each point:

   a. along the positive x-axis.
   b. along the positive y-axis.
   c. along the negative x-axis.
   d. along the negative y-axis.
   e. at a point in the first quadrant in the x-y plane, making an angle of 45° with both the positive-x and positive-y axes.

3. An electric dipole of dipole moment $\vec{p} = p\hat{x}$, with $p = 5.0 \times 10^{-15}\,\mathrm{C\,m}$, is placed in a uniform electric field $\vec{E} = E_0\hat{x}$ with $E_0 = 4.0 \times 10^6\,\mathrm{N/C}$.

   a. What torque is exerted on the dipole by the field?

b. What is the potential energy of the dipole in this field?

c. If you (with tweezers?) rotated this dipole so that it aligned perpendicular to the field, $\vec{p} = p\hat{x}$, how much work would you (not the field!) have to do?

4. Given a finite-sized dipole with a charge of -2.5×10^5 C at one end-point and a charge of $+2.5 \times 10^5$ C at the other end-point, and a distance between the two points of 0.0010 m, and with the dipole oriented at 40.0° to a uniform external electric field, find:

   a. the dipole moment.

   b. with external electric field strength equal to $7.5\,\mathrm{N/C}\,\hat{x}$, find the potential energy of the dipole in this field.

   c. the torque on the dipole due to the field.

   d. the work done by the field on the dipole in rotating it from 40.0° with the field to its equilibrium orientation.

Brief Answers:

1. a. (i) 0; (ii) $-E_0 p$; (iii) 0; (iv) $+E_0 p$. Order of decreasing energy: (iv), (i) and (iii), (ii).

   b. The lowest energy alignment: (ii) $\vec{p} = p\hat{y}$.

2. a. $\vec{E} = -9\hat{y}\,\mathrm{N/C}$.

   b. $\vec{E} = 18\hat{y}\,\mathrm{N/C}$.

   c. $\vec{E} = -9\hat{y}\,\mathrm{N/C}$.

   d. $\vec{E} = 18\hat{y}\,\mathrm{N/C}$.

   e. $\vec{E} = (13.5\hat{x} + 4.5\hat{y})\,\mathrm{N/C}$.

3. a. Zero.

   b. -2.0×10^{-8} joules.

   c. 2.0×10^{-8} joules

4. a. 250 C m; directed from the negative to the positive charge.

   b. -1436 J.

   c. -1205 N m; into the paper, or 1205 N m out of the paper.

   d. $+439$ J.

MODEL EXAM

1. See Output Skills K1-K2 in this module's *ID Sheet*. One or both of these skills may be on the actual exam.

2. Given a finite-sized dipole with a charge of -2.5×10^5 C at one end-point and a charge of $+2.5 \times 10^5$ C at the other end-point, and a distance between the two points of 0.0010 m, and with the dipole oriented at 40.0° to a uniform external electric field, find:

   a. the dipole moment.

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   c. the torque on the dipole due to the field.

   d. the work done by the field on the dipole in rotating it from 40.0° with the field to its equilibrium orientation.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 4.